International Journal of Theoretical Physics, Vol. 35, No. 9, 1996

Dirac–Hestenes Spinor Fields on Riemann–Cartan Manifolds

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Received January 9, 1996

In this paper we study Dirac–Hestenes spinor fields (DHSF) on a four-dimensional Riemann–Cartan spacetime (RCST). We prove that these fields must be defined as certain equivalence classes of even sections of the Clifford bundle (over the RCST), thereby being certain particular sections of a new bundle named the spin-Clifford bundle (SCB). The conditions for the existence of the SCB are studied and are shown to be equivalent to Geroch's theorem concerning the existence of spinor structures in a Lorentzian spacetime. We introduce also the covariant and algebraic Dirac spinor fields and compare these with DHSF, showing that all three kinds of spinor fields contain the same mathematical and physical information. We clarify also the notion of (Crumeyrolle's) amorphous spinors (Dirac–Kähler spinor fields are of this type), showing that they cannot be used to describe fermionic fields. We develop a rigorous theory for the covariant derivatives of Clifford fields (sections of the Clifford bundle, CB) and of Dirac–Hestenes spinor fields. We show how to generalize the original Dirac–Hestenes equation in Minkowski spacetime for the case of RCST. Our results are obtained from a variational principle formulated through the multiform derivative approach to Lagrangian field theory in the Clifford bundle.

1. INTRODUCTION

In the following we study the theory of Dirac–Hestenes spinor fields (DHSF) and the theory of their covariant derivatives on a Riemann-Cartan spacetime (RCST). We also show how to generalize the so-called Dirac–Hestenes equation—originally introduced in Hestenes (1967, 1976) for the formulation of the Dirac theory of the electron using the spacetime algebra $\mathcal{C}\ell_{1,3}$ in Minkowski spacetime—for an arbitrary Riemann–Cartan spacetime.

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0020-7748/96/0900-1849$09.50/0

We use an approach based on the multiform derivative formulation of Lagrangian field theory to obtain the above results. They are important for the study of spinor fields in gravitational theory and are essential for an understanding of the relationship between the Maxwell and Dirac theories and quantum mechanics (Vaz and Rodrigues, 1993a, 1995).

In order to achieve our goals we start by clarifying many misconceptions concerning the usual presentation of the theory of covariant, algebraic, and Dirac–Hestenes spinors. Section 2 is dedicated to this subject and we believe that it improves over other presentations (e.g., Vaz and Rodrigues, 1993a; Figueiredo *et al.*, 1990a,b; Rodrigues and Oliveira, 1990; Rodrigues and Figueiredo, 1990; Lounesto, 1993, 1994; Benn and Tucker, 1987; Blau, 1985), introducing a new and important fact, namely that all kind of spinors referred to above must be defined as special equivalence classes in appropriate Clifford algebras. The hidden geometrical meaning of the covariant Dirac spinor is disclosed and the physical and geometrical meaning of the famous Fierz identities (Rodrigues and Figueiredo, 1990; Lounesto, 1993; Fierz, 1937; Crawford, 1985) becomes obvious.

In Section 3 we study the Clifford bundle of a Riemann–Cartan spacetime (de Souza and Rodrigues, 1994) and its irreducible module representations. This permits us to define Dirac–Hestenes spinor fields (DHSF) as certain equivalence classes of even sections of the Clifford bundle. DHSF are then naturally identified with sections of a new bundle which we call the spin-Clifford bundle.

We discuss also the concept of amorphous spinor fields (ASF)—a name introduced by Crumeyrolle (1991). The so-called Dirac–Kähler spinors (Kähler, 1962) discussed by Graf (1978) and used in presentations of field theories in the lattice (Becher, 1981; Becher and Joos, 1982) are examples of ASF. We prove that they cannot be used to describe fermion fields because they cannot be used to properly formulate the Fierz identities.

In Section 4 we show how the Clifford and spin-Clifford bundle techniques permit us to give a simple presentation of the concept of covariant derivative for Clifford fields, algebraic Dirac spinor fields, and the DHSF. We show that our elegant theory agrees with the standard one developed for the so-called covariant Dirac spinor fields as developed, e.g., in Lichnerowicz (1964, 1984).

In Section 5 we introduce the concepts of Dirac and spin-Dirac operators acting respectively on sections of the Clifford and spin-Clifford bundles. We show how to use the spin-Dirac operator on the representatives of DHSF on the Clifford bundle.

In Section 6 we present the multiform derivative approach to Lagrangian field theory and derive the Dirac–Hestenes equation on RCST (Choquet-Bruhat *et al.*, 1982). We compare our results with others for the covariant

Dirac spinor field (Rodrigues *et al.*, 1994; Hehl and Datta, 1971) and also for Dirac–Kähler fields (Kähler, 1962; Graf, 1978; Ivanenko and Obukhov, 1985).

Finally in Section 7 we present our conclusions.

2. COVARIANT, ALGEBRAIC, AND DIRAC–HESTENES SPINORS

2.1. Some General Features about Clifford Algebras

In this section we fix our notations and introduce the main ideas concerning the theory of Clifford algebras necessary for the intelligibility of the paper. We follow with minor modifications the conventions used in Rodrigues and Figueiredo (1990) and Lounesto (1993).

2.1.1. Formal Definition of the Clifford Algebra $\mathcal{C}\ell(V, Q)$

Let K be a field, char $K \neq 2$,[3] V a vector space of finite dimension n over K, and Q a nondegenerate quadratic form over V. Denote by

$$\mathbf{x}\cdot\mathbf{y} = \tfrac{1}{2}(Q(\mathbf{x} + \mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})) \tag{1}$$

The associated *symmetric* bilinear form on V and define the *left contraction* $\rfloor\colon \wedge V \times \wedge V \to \wedge V$ and the *right contraction* $\lfloor\colon \wedge V \times \wedge V \to \wedge V$ by the following rules:

1. $\mathbf{x} \rfloor \mathbf{y} = \mathbf{x}\cdot\mathbf{y}$
   $\mathbf{x} \lfloor \mathbf{y} = \mathbf{x}\cdot\mathbf{y}$
2. $\mathbf{x} \rfloor (u \wedge v) = (\mathbf{x} \rfloor u) \wedge v + \hat{u} \wedge (\mathbf{x} \rfloor v)$
   $(u \wedge v) \lfloor \mathbf{x} = u \wedge (v \lfloor \mathbf{x}) + (u \lfloor \mathbf{x}) \wedge \hat{v}$
3. $(u \wedge v) \rfloor w = u \rfloor (u \rfloor w)$
   $u \lfloor (v \wedge w) = (u \lfloor v) \lfloor w$

Here $\mathbf{x}, \mathbf{y} \in V$, $u, v, w \in \wedge V$, and ^ is the grade involution in the algebra $\wedge V$. The notation $\mathbf{a}\cdot\mathbf{b}$ will be used for contractions when it is clear from the context which factor is the contractor and which factor is being contracted. When just one of the factors is homogeneous, it is understood to be the contractor. When both factors are homogeneous, we agree that the one with lower degree is the contractor, so that for $\mathbf{a} \in \wedge^r V$ and $\mathbf{b} \in \wedge^s V$, we have $\mathbf{a}\cdot\mathbf{b} = \mathbf{a} \rfloor \mathbf{b}$ if $r \leq s$ and $\mathbf{a}\cdot\mathbf{b} = \mathbf{a} \lfloor \mathbf{b}$ if $r \geq s$.

Define the (Clifford) product of $\mathbf{x} \in V$ and $u \in \wedge V$ by

$$\mathbf{x}u = \mathbf{x} \wedge u + \mathbf{x} \rfloor u \tag{2}$$

[3] In our applications in this paper, K will be R or C, respectively, the real or complex field. The quaternion ring will be denoted by H.

and extend this product by linearity and associativity to all of $\wedge V$. This provides $\wedge V$ with a new product, and provided with this new product $\wedge V$ becomes isomorphic to the *Clifford algebra* $\mathcal{C}\ell(V, Q)$.

We recall that $\wedge V = T(V)/I$, where $T(V)$ is the tensor algebra of V and $I \subset T(V)$ is the bilateral ideal generated by the elements of the form $\mathbf{x} \otimes \mathbf{x}$, $\mathbf{x} \in V$. It can also be shown that the Clifford algebra of (V, Q) is $\mathcal{C}\ell(V, Q) = T(V)/I_Q$, where I_Q is the bilateral ideal generated by the elements of the form $\mathbf{x} \otimes \mathbf{x} - Q(\mathbf{x})$, $\mathbf{x} \in V$. The Clifford algebra so constructed is an associative algebra with unity. Since K is a field, the space V is naturally imbedded in $\mathcal{C}\ell(V, Q)$,

$$V \overset{i}{\hookrightarrow} T(V) \overset{j}{\rightarrow} T(V)/I_Q = \mathcal{C}\ell(V, Q)$$

$$I_Q = j \circ i \qquad \text{and} \qquad V \equiv i_Q(V) \subset \mathcal{C}\ell(V, Q) \tag{3}$$

Let $\mathcal{C}\ell^+(V, Q)$ [resp., $\mathcal{C}\ell^-(V, Q)$] be the j-image of $\oplus_{i=0}^{\infty} T^{2i}(V)$ [resp., $\oplus_{i=0}^{\infty} T^{2i+1}(V)$] in $\mathcal{C}\ell(V, Q)$. The elements of $\mathcal{C}\ell^+(V, Q)$ form a subalgebra of $\mathcal{C}\ell(V, Q)$ called the even subalgebra of $\mathcal{C}\ell(V, Q)$.

$\mathcal{C}\ell(V, Q)$ has the following property: If A is an associative K-algebra with unity, then all linear mappings $\rho: V \rightarrow A$ such that $(\rho(x))^2 = Q(x)$, $x \in V$, can be extended in a unique way to an algebra homomorphism $\rho: \mathcal{C}\ell(V, Q) \rightarrow A$.

In $\mathcal{C}\ell(V, Q)$ there exist three linear mappings which are quite natural. They are extensions of the following mappings:

Main Involution. An automorphism $\hat{}: \mathcal{C}\ell(V, Q) \rightarrow \mathcal{C}\ell(V, Q)$, extension of $\alpha: V \rightarrow T(V)/I_Q$, $\alpha(x) = -i_Q(x) = -x$, $\forall x \in V$.

Reversion. An antiautomorphism $\tilde{}: \mathcal{C}\ell(V, Q) \rightarrow \mathcal{C}\ell(V, Q)$, extension of ${}^t: T^r(V) \rightarrow T^r(V)$; $T^r(V) \ni x = x_{i_1} \otimes \cdots \otimes x_{i_r} \mapsto x^t = x_{i_r} \otimes \cdots \otimes x_{i_1}$.

Conjugation. $\bar{}: \mathcal{C}\ell(V, Q) \rightarrow \mathcal{C}\ell(V, Q)$, defined by the composition of the main involution $\hat{}$ with the reversion $\tilde{}$, i.e., if $x \in \mathcal{C}\ell(V, Q)$, then $\bar{x} = (\hat{x})^\sim = (\tilde{x})^\wedge$.

$\mathcal{C}\ell(V, Q)$ can be described through its generators, i.e., if $\Sigma = \{E_i\}$ $(i = 1, 2, \ldots, n)$ is a Q-orthonormal basis of V, then $\mathcal{C}\ell(V, Q)$ is generated by 1 and the E_i are subject to the conditions

$$\begin{aligned} E_i E_i &= Q(E_i) \\ E_i E_j + E_j E_i &= 0, \qquad i \neq j; \quad i, j = 1, 2, \ldots, n \\ E_1 E_2 \cdots E_n &\neq \pm 1 \end{aligned} \tag{4}$$

2.1.2. *The Real Clifford Algebra* $\mathcal{C}\ell_{p,q}$

Let $\mathbb{R}^{p,q}$ be a real vector space of dimension $n = p + q$ endowed with a nondegenerate metric $g: \mathbb{R}^{p,q} \times \mathbb{R}^{p,q} \to \mathbb{R}$. Let $\Sigma = \{E_i\}$ $(i = 1, 2, \ldots, n)$ be an orthonormal basis of $\mathbb{R}^{p,q}$,

$$g(E_i, E_j) = g_{ij} = g_{ji} = \begin{cases} +1, & i = j = 1, 2, \ldots, p \\ -1, & i = j = p + 1, \ldots, p + q = n \\ 0, & i \neq j \end{cases} \quad (5)$$

The Clifford algebra $\mathcal{C}\ell_{p,q} = \mathcal{C}\ell(\mathbb{R}^{p,q}, Q)$ is the Clifford algebra over $\mathbb{R}$, generated by 1 and the $\{E_i\}$ $(i = 1, 2, \ldots, n)$ such that $E_i^2 = Q(E_i) = g(E_i, E_i)$, $E_iE_j = -E_jE_i$ $(i \neq j)$, and (Ablamowicz *et al.*, 1991) $E_1E_2 \cdots E_n \neq \pm 1$. The algebra $\mathcal{C}\ell_{p,q}$ is obviously of dimension 2^n and as a vector space it is the direct sum of vector spaces $\wedge^k \mathbb{R}^{p,q}$ of dimensions $\binom{n}{k}$, $0 \leq k \leq n$. The canonical basis of $\wedge^k \mathbb{R}^{p,q}$ is given by the elements $e_A = E_{\alpha_1} \cdots E_{\alpha_k}$, $1 \leq \alpha_1 < \cdots < \alpha_k \leq n$. The element $c_J = E_1 \cdots E_n \in \wedge^n \mathbb{R}^{p,q}$ commutes (n odd) or anticommutes (n even) with all vectors $E_1, \ldots, E_n \in \wedge^1\mathbb{R}^{p,q} \equiv \mathbb{R}^{p,q}$. The center of $\mathcal{C}\ell_{p,q}$ is $\wedge^0\mathbb{R}^{p,q} \equiv \mathbb{R}$ if n is even and it is the direct sum $\wedge^0\mathbb{R}^{p,q} \oplus \wedge^n\mathbb{R}^{p,q}$ if n is odd.

All Clifford algebras are semisimple. If $p + q = n$ is even, $\mathcal{C}\ell_{p,q}$ is simple, and if $p + q = n$ is odd, we have the following possibilities:

1. $\mathcal{C}\ell_{p,q}$ is simple $\leftrightarrow c_J^2 = -1 \leftrightarrow p - q \neq 1 \pmod 4 \leftrightarrow$ center of $\mathcal{C}\ell_{p,q}$ is isomorphic to $\mathbb{C}$.
2. $\mathcal{C}\ell_{p,q}$ is not simple (but is a direct sum of two simple algebras) $\leftrightarrow$ $c_J^2 = +1 \leftrightarrow p - q = 1 \pmod 4 \leftrightarrow$ center of $\mathcal{C}\ell_{p,q}$ is isomorphic to $\mathbb{R} \oplus \mathbb{R}$.

All these semisimple algebras are direct sums of two simple algebras.

If A is an associative algebra on the field K, $K \subseteq A$, and if E is a vector space, a homomorphism ρ from A to End E (End E is the endomorphism algebra of E) which maps the unit element of A to Id_E is a called a *representation* of A in E. The dimension E is called the degree of the representation. The addition in E together with the mapping $A \times E \to E$, $(a, x) \mapsto \rho(a)x$ turns E into an A-module, the *representation module*.

Conversely, A being an algebra over K and E being an A-module, E is a vector space over K and if $a \in A$, the mapping $\gamma: A \to \gamma_a$ with $\gamma_a(x) = ax$, $x \in E$, is a homomorphism $A \to$ End E, and so it is a representation of A in E. The study of A-modules is then equivalent to the study of the representations of A. A representation ρ is *faithful* if its kernel is zero, i.e., $\rho(a)x = 0$, $\forall x \in E \Rightarrow a = 0$. The kernel of ρ is also known as the annihilator of its module. ρ is said to be *simple* or irreducible if the only invariant subspaces of $\rho(a)$, $\forall a \in A$, are E and $\{0\}$. Then the representation module is also simple, meaning that it has no proper submodule. ρ is said to be *semisimple* if it is the direct sum of simple modules, and in this case E is the direct sum of subspaces which are globally invariant under $\rho(a)$, $\forall a \in A$. When no confusion arises $\rho(a)x$ will be denoted by $a \bullet x$, $a * x$, or ax. Two A-modules E and E' (with the exterior multiplication being denoted respectively by • and $*$) are *isomorphic* if there exists a bijection $\varphi: E \to E'$ such that

$$\varphi(x + y) = \varphi(x) + \varphi(y), \qquad \forall x, y \in E$$
$$\varphi(a \bullet x) = a * \varphi(x), \qquad \forall a \in A \tag{6}$$

and we say that representations ρ and ρ′ of A are equivalent if their modules are isomorphic. This implies the existence of a K-linear isomorphism $\varphi: E \to E'$ such that $\varphi \circ \rho(a) = \rho'(a) \circ \varphi$, $\forall a \in A$, or $\rho'(a) = \varphi \circ \rho(a) \circ \varphi^{-1}$. If $\dim E = n$, then $\dim E' = n$. We shall need the following result.

Wedderburn Theorem (Porteous, 1969). If A is a simple algebra, then A is equivalent to $F(m)$, where $F(m)$ is a matrix algebra with entries in F, F is a division algebra, and m and F are unique (modulo isomorphisms).

2.2. Minimal Left Ideals of $\mathcal{C}\ell_{p,q}$

The minimal left (resp., right) ideals of a semisimple algebra A are of the type Ae (resp., eA), where e is a primitive idempotent of A, i.e., $e^2 = e$ and e cannot be written as a sum of two nonzero annihilating (or orthogonal) idempotents, i.e., $e \neq e_1 + e_2$, where $e_1 e_2 = e_2 e_1 = 0$, $e_1^2 = e_1$, $e_2^2 = e_2$.

Theorem. The maximum number of pairwise annihilating idempotents in $F(m)$ is m.

The decomposition of $\mathcal{C}\ell_{p,q}$ into minimal ideals is then characterized by a spectral set $\{e_{pq,i}\}$ of idempotents of $\mathcal{C}\ell_{p,q}$ satisfying (i) $\sum_i e_{pq,i} = 1$; (ii) $e_{pq,i} e_{pq,j} = \delta_{ij} e_{pq,i}$; (iii) rank of $e_{pq,i}$ is minimal $\neq 0$, i.e., $e_{pq,i}$ is primitive ($i = 1, 2, \ldots, m$).

By rank of $e_{pq,i}$ we mean the rank of the $\wedge\mathbb{R}^{p+q}$-morphism $e_{pq,i}$: $\psi \mapsto \psi e_{pq,i}$ and $\wedge\mathbb{R}^{p,q} = \bigoplus_{k=0}^{n} \wedge^k(\mathbb{R}^{p,q})$ is the exterior algebra of $\mathbb{R}^{p,q}$. Then $\mathcal{C}\ell_{p,q} = \sum_i I^i_{p,q}$, $I^i_{p,q} = \mathcal{C}\ell_{p,q} e_{pq,i}$, and $\psi \in I^i_{p,q}$ is such that $\psi e_{pq,i} = \psi$. Conversely any element $\psi \in I^i_{p,q}$ can be characterized by an idempotent $e_{pq,i}$ of minimal rank $\neq 0$ with $\psi e_{pq,i} = \psi$. We have the following result.

Theorem (Lounesto, 1981). A minimal left ideal of $\mathcal{C}\ell_{p,q}$ is of the type $I_{p,q} = \mathcal{C}\ell_{p,q} e_{pq}$, where $e_{pq} = \frac{1}{2}(1 + e_{\alpha_1}) \cdots \frac{1}{2}(1 + e_{\alpha_k})$ is a primitive idempotent of $\mathcal{C}\ell_{p,q}$ and $e_{\alpha_1}, \ldots, e_{\alpha_k}$ are commuting elements of the canonical basis of $\mathcal{C}\ell_{p,q}$ such that $(e_{\alpha_i})^2 = 1$ ($i = 1, 2, \ldots, k$) that generate a group of order

2^k, $k = q - r_{q-p}$, and r_i are the Radon–Hurwitz numbers, defined by the recurrence formula $r_{i+8} = r_i + 4$ and

i	0	1	2	3	4	5	6	7
r_i	0	1	2	2	3	3	3	3

If we have a linear mapping L_a: $\mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{p,q}$, $L_a(x) = ax$, $x \in \mathcal{C}\ell_{p,q}$, $a \in \mathcal{C}\ell_{p,q}$, then since $I_{p,q}$ is invariant under left multiplication with arbitrary elements of $\mathcal{C}\ell_{p,q}$, we can consider $L_a|_{I_{p,q}}$: $I_{p,q} \to I_{p,q}$ and taking into account the Wedderburn theorem we have the following result.

Theorem. If $p + q = n$ is even or odd with $p - q \neq 1 \pmod 4$, then

$$\mathcal{C}\ell_{p,q} \simeq \mathrm{End}_F(I_{p,q}) \simeq F(m)$$

where $F = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, $\mathrm{End}_F(I_{p,q})$ is the algebra of linear transformations in $I_{p,q}$ over the field F, $m = \dim_F(I_{p,q})$, and $F \simeq eF(m)e$, e being the representation of e_{pq} in $F(m)$.

If $p + q = n$ is odd, with $p - q = 1 \pmod 4$, then

$$\mathcal{C}\ell_{p,q} = \mathrm{End}_F(I_{p,q}) \simeq F(m) \oplus F(m)$$

and $m = \dim_F(I_{p,q})$ and $e_{pq}\mathcal{C}\ell_{p,q}e_{pq} \simeq \mathbb{R} \oplus \mathbb{R}$ or $\mathbb{H} \oplus \mathbb{H}$.

Observe that F is the set

$$F = \{T \in \mathrm{End}_F(I_{p,q}),\ TL_a = L_aT,\ \forall a \in \mathcal{C}\ell_{p,q}\}$$

Periodicity Theorem (Porteous, 1969). For $n = p + q \geq 0$ there exist the following isomorphisms:

$$\mathcal{C}\ell_{n+8} \simeq \mathcal{C}\ell_{n,0} \otimes \mathcal{C}\ell_{8,0}, \qquad \mathcal{C}\ell_{0,n+8} \simeq \mathcal{C}\ell_{0,n} \otimes \mathcal{C}\ell_{0,8} \tag{7}$$
$$\mathcal{C}\ell_{p+8,q} \simeq \mathcal{C}\ell_{p,q} \otimes \mathcal{C}\ell_{8,0}, \qquad \mathcal{C}\ell_{p,q+8} \simeq \mathcal{C}\ell_{p,q} \otimes \mathcal{C}\ell_{0,8}$$

We can find, e.g., in Porteous (1969) and Figueiredo *et al.* (1990a,b) tables giving the representations of all algebras $\mathcal{C}\ell_{p,q}$ as matrix algebras. For what follows we need

$$\begin{aligned} \text{complex numbers} \quad & \mathcal{C}\ell_{0,1} \simeq \mathbb{C} \\ \text{quarternions} \quad & \mathcal{C}\ell_{0,2} \simeq \mathbb{H} \\ \text{Pauli algebra} \quad & \mathcal{C}\ell_{3,0} \simeq M_2(\mathbb{C}) \\ \text{spacetime algebra} \quad & \mathcal{C}\ell_{1,3} \simeq M_2(\mathbb{H}) \\ \text{Majorana algebra} \quad & \mathcal{C}\ell_{3,1} \simeq M_4(\mathbb{R}) \\ \text{Dirac algebra} \quad & \mathcal{C}\ell_{4,1} \simeq M_4(\mathbb{C}) \end{aligned} \tag{8}$$

We also need the following result.

Proposition. $\mathcal{C}\ell^+_{p,q} = \mathcal{C}\ell_{q,p-1}$ for $p > 1$ and $\mathcal{C}\ell^+_{p,q} = \mathcal{C}\ell_{p,q-1}$ for $q > 1$.

From the above proposition we get the following particular results that we shall need later:

$$\mathcal{C}\ell^+_{1,3} \simeq \mathcal{C}\ell^+_{3,1} = \mathcal{C}\ell_{3,0}, \qquad \mathcal{C}\ell^+_{4,1} \simeq \mathcal{C}\ell_{1,3} \tag{9}$$

$$\mathcal{C}\ell_{4,1} \simeq \mathbb{C} \otimes \mathcal{C}\ell_{3,1}, \qquad \mathcal{C}\ell_{4,1} \simeq \mathbb{C} \otimes \mathcal{C}\ell_{1,3} \tag{10}$$

which means that the Dirac algebra is the complexification of both the spacetime or the Majorana algebras.

Right Linear Structure for $I_{p,q}$. We can give to the ideal $I_{p,q} = \mathcal{C}\ell_{p,q}e$ (resp. $I_{pq} = e\mathcal{C}\ell_{pq}$) a right (resp. left) linear structure over the field $F(\mathcal{C}\ell_{p,q} \simeq F(m)$ or $\mathcal{C}\ell_{p,q} \simeq F(m) \oplus F(m))$. A right linear structure, e.g., consists of an additive group (which is $I_{p,q}$) and the mapping

$$I \times F \to I; \qquad (\psi, T) \mapsto \psi T$$

such that the usual axioms of a linear vector space structure are valid, e.g., we have[4] $(\psi T)T' = \psi(TT')$.

From the above discussion it is clear that the minimal (left or right) ideals of $\mathcal{C}\ell_{p,q}$ are representation modules of $\mathcal{C}\ell_{p,q}$. In order to investigate the equivalence of these representations we must introduce some groups that are subsets of $\mathcal{C}\ell_{p,q}$. As we shall see, this is the key for the definition of algebraic and Dirac–Hestenes spinors.

2.3. The Groups: $\mathcal{C}\ell^*_{p,q}$, Clifford, Pinor, and Spinor

The set of the invertible elements of $\mathcal{C}\ell_{p,q}$ constitutes a non-Abelian group which we denote by $\mathcal{C}\ell^*_{p,q}$. It acts naturally on $\mathcal{C}\ell_{p,q}$ as an algebra homomorphism through its adjoint representation

$$\text{Ad}\colon\ \mathcal{C}\ell^*_{p,q} \to \text{Aut}(\mathcal{C}\ell_{p,q}); \qquad u \mapsto \text{Ad}_u, \quad \text{with} \quad \text{Ad}_u(x) = uxu^{-1} \tag{11}$$

The Clifford–Lipschitz group is the set

$$\mathbf{\Gamma}_{p,q} = \{u \in \mathcal{C}\ell^*_{p,q} \mid \forall x \in \mathbb{R}^{p,q},\ ux\hat{u}^{-1} \in \mathbb{R}^{p,q}\} \tag{12}$$

The set $\mathbf{\Gamma}^+_{p,q} = \mathbf{\Gamma}_{p,q} \cap \mathcal{C}\ell^+_{p,q}$ is called special Clifford–Lipschitz group.

Let $N\colon \mathcal{C}\ell_{p,q} \to \mathcal{C}\ell_{p,q}$, $N(x) = \langle \tilde{x}x\rangle_0$ ($\langle\cdot\rangle_0$ means the scalar part of the Clifford number). We define further:

The *Pinor group* $\mathbf{Pin}(p, q)$ is the subgroup of $\mathbf{\Gamma}_{p,q}$ such that

$$\mathbf{Pin}(p, q) = \{u \in \mathbf{\Gamma}_{p,q} \mid N(u) = \pm 1\} \tag{13}$$

The *Spin group* $\mathbf{Spin}(p, q)$ is the set

[4] For $\mathcal{C}\ell_{3,0}$, $I = \mathcal{C}\ell_{3,0}\frac{1}{2}(1 + \sigma_3)$ is a minimal left ideal. In this case it is also possible to give a left linear structure for this ideal. See Vaz and Rodrigues (1993a) and Figueiredo *et al.* (1990a).

$$\mathbf{Spin}(p, q) = \{u \in \Gamma^{+}_{p,q} | N(u) = \pm 1\} \tag{14}$$

The $\mathbf{Spin}_{+}(p, q)$ group is the set

$$\mathbf{Spin}_{+}(p, q) = \{u \in \Gamma^{+}_{p,q} | N(u) = +1\} \tag{15}$$

Theorem. $\mathrm{Ad}_{|\mathbf{Pin}(p,q)}$: $\mathbf{Pin}(p, q) \to \mathrm{O}(p, q)$ is onto with kernel $\mathbf{Z}_2$. $\mathrm{Ad}_{|\mathbf{Spin}(p,q)}$: $\mathbf{Spin}(p, q) \to \mathrm{SO}(p, q)$ is onto with kernel $\mathbf{Z}_2$.

$\mathrm{O}(p, q)$ is the pseudoorthogonal group of the vector space $\mathbb{R}^{p,q}$, $\mathrm{SO}(p, q)$ is the special pseudoorthogonal group of $\mathbb{R}^{p,q}$. We also denote by $\mathrm{SO}_{+}(p, q)$ the connected component of $\mathrm{SO}(p, q)$. $\mathbf{Spin}_{+}(p, q)$ is connected for all pairs (p, q) with the exception of $\mathbf{Spin}_{+}(1, 0) \simeq \mathbf{Spin}_{+}(0, 1) \simeq \{\pm 1\}$ and $\mathbf{Spin}_{+}(1, 1)$. We have

$$\mathrm{O}(p, q) = \frac{\mathbf{Pin}(p, q)}{\mathbf{Z}_2}, \qquad \mathrm{SO}(p, q) = \frac{\mathbf{Spin}(p, q)}{\mathbf{Z}_2}, \qquad \mathrm{SO}_{+}(p, q) = \frac{\mathbf{Spin}_{+}(p, q)}{\mathbf{Z}_2}$$

In the following the group homomorphism between $\mathbf{Spin}_{+}(p, q)$ and $\mathrm{SO}_{+}(p, q)$ will be denoted

$$\mathcal{H}: \mathbf{Spin}_{+}(p, q) \to \mathrm{SO}_{+}(p, q) \tag{16}$$

We also need the following important result:

Theorem (Lounesto, 1981). For $p + q \leq 5$, $\mathbf{Spin}_{+}(p, q) = \{u \in \mathcal{C}\ell^{+}_{p,q} | u\tilde{u} = 1\}$.

Lie Algebra of $\mathbf{Spin}_{+}(1, 3)$. It can be shown that for each $u \in \mathbf{Spin}_{+}(1, 3)$ one has

$$u = \pm e^{F}, \qquad F \in \overset{2}{\wedge}\mathbb{R}^{1,3} \subset \mathcal{C}\ell_{1,3} \tag{17}$$

and F can be chosen in such a way as to have a positive sign in (17) except in the particular case $F^2 = 0$ when $u = -e^{F}$. From (17) it follows immediately that the Lie algebra of $\mathbf{Spin}_{+}(1, 3)$ is generated by the bivectors $F \in \wedge^2\mathbb{R}^{1,3} \subset \mathcal{C}\ell_{1,3}$ through the commutator product.

2.4. Geometrical and Algebraic Equivalence of the Representation Modules $I_{p,q}$ of Simple Clifford Algebras $\mathcal{C}\ell_{p,q}$

Recall that $\mathcal{C}\ell_{p,q}$ is a ring. We already said that the minimal lateral ideals of $\mathcal{C}\ell_{p,q}$ are of the form $I_{p,q} = \mathcal{C}\ell_{p,q}e_{pq}$ (or $e_{pq}\mathcal{C}\ell_{p,q}$), where e_{pq} is a primitive idempotent. Obviously the minimal lateral ideals are modules over the ring $\mathcal{C}\ell_{p,q}$; they are representation modules. According to the discussion of Section 2.1, given two ideals $I_{p,q} = \mathcal{C}\ell_{p,q}e_{pq}$ and $I'_{p,q} = \mathcal{C}\ell_{p,q}e'_{pq}$ they are by definition isomorphic if there exists a bijection φ: $I_{p,q} \to I'_{p,q}$ such that

$$\varphi(\psi_1 + \psi_2) = \varphi(\psi_1) + \varphi(\psi_2); \qquad \varphi(a\psi) = a\varphi(\psi)$$
$$\forall a \in \mathcal{C}\ell_{p,q}, \qquad \forall \psi_1, \psi_2 \in I_{p,q} \tag{18}$$

Recalling the Noether–Skolem theorem, which says that all automorphisms of a simple algebra are inner automorphisms, we have the following result.

Theorem. When $\mathcal{C}\ell_{p,q}$ is simple, its automorphisms are given by inner automorphisms $x \mapsto uxu^{-1}$, $x \in \mathcal{C}\ell_{p,q}$, $u \in \mathcal{C}\ell^*_{p,q}$.

We also have the following result.

Proposition. When $\mathcal{C}\ell_{p,q}$ is simple, all its finite-dimensional irreducible representations are equivalent (i.e., isomorphic) under inner automorphisms.

We quote also the following result.

Theorem (Crumeyrolle, 1991). $I_{p,q}$ and $I'_{p,q}$ are isomorphic if and only if $I'_{p,q} = I_{p,q}X$ for nonzero $X \in I'_{p,q}$.

We are thus led to the following definitions:

1. The ideals $I_{p,q} = \mathcal{C}\ell_{p,q}e_{pq}$ and $I'_{p,q} = \mathcal{C}\ell_{p,q}e'_{pq}$ are said to be *geometrically equivalent* if, for some $u \in \mathbf{\Gamma}_{p,q}$,

$$e'_{pq} = ue_{pq}u^{-1} \tag{19}$$

2. $I_{p,q}$ and $I'_{p,q}$ are said to be *algebraically equivalent* if

$$e'_{pq} = ue_{pq}u^{-1} \tag{20}$$

for some $u \in \mathcal{C}\ell^*_{p,q}$, but $u \notin \mathbf{\Gamma}_{p,q}$.

It is now time to specialize the above results for $\mathcal{C}\ell_{1,3} \simeq M_2(\mathbb{H})$ and to find a relationship between the Dirac algebra $\mathcal{C}\ell_{4,1} \simeq M_4(\mathbb{C})$ and $\mathcal{C}\ell_{1,3}$ and their respective minimal ideals.

Let $\Sigma_0 = \{E_0, E_1, E_2, E_3\}$ be an orthogonal basis of $\mathbb{R}^{1,3} \subset \mathcal{C}\ell_{1,3}$, $E_\mu E_\nu + E_\nu E_\mu = 2\eta_{\mu\nu}$, $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$. Then, the elements

$$e = \tfrac{1}{2}(1 + E_0), \qquad e' = \tfrac{1}{2}(1 + E_3E_0), \qquad e'' = \tfrac{1}{2}(1 + E_1E_2E_3) \tag{21}$$

are easily verified to be primitive idempotents of $\mathcal{C}\ell_{1,3}$. The minimal left ideals $I = \mathcal{C}\ell_{1,3}e$, $I' = \mathcal{C}\ell_{1,3}e'$, and $I'' = \mathcal{C}\ell_{1,3}e''$ are right two-dimensional linear spaces over the quaternion field (e.g., $\mathbb{H}e = e\mathbb{H} = e\mathcal{C}\ell_{1,3}e$). According to the definition 2 above, these ideals are algebraically equivalent. For example, $e' = ueu^{-1}$, with $u = (1 + E_3) \notin \mathbf{\Gamma}_{1,3}$.

The elements $\Phi \in \mathcal{C}\ell_{1,3}\frac{1}{2}(1 + E_0)$ will be called *mother* spinors (Lounesto, 1993a,b). We can show (Figueiredo *et al.*, 1990a) that each Φ can be written

$$\Phi = \psi_1 e + \psi_2 E_3 E_1 e + \psi_3 E_3 E_0 e + \psi_4 E_1 E_0 e = \sum_i \psi_i s_i \tag{22}$$

$$s_1 = e, \qquad s_2 = E_3 E_1 e, \qquad s_3 = E_3 E_0 e, \qquad s_4 = E_1 E_0 e \tag{23}$$

and where the ψ_i are formally complex numbers, i.e., each $\psi_i = (a_i + b_i E_2 E_1)$ with $a_i, b_i \in \mathbb{R}$.

We recall that $\mathbf{Pin}(1, 3)/\mathbf{Z}_2 \simeq \mathrm{O}(1, 3)$, $\mathbf{Spin}(1, 3)/\mathbf{Z}_2 \simeq \mathrm{SO}(1, 3)$, $\mathbf{Spin}_+(1, 3)/\mathbf{Z}_2 \simeq \mathrm{SO}_+(1, 3)$, and $\mathbf{Spin}_+(1, 3) \simeq \mathrm{SL}(2, \mathbb{C})$ the universal covering group of $\mathcal{L}_+^\uparrow \equiv \mathrm{SO}_+(1, 3)$, the restricted Lorentz group.

In order to determine the relation between $\mathcal{C}\ell_{4,1}$ and $\mathcal{C}\ell_{1,3}$ we proceed as follows: let $\{F_0, F_1, F_2, F_3, F_4\}$ be an orthogonal basis of $\mathcal{C}\ell_{4,1}$ with $-F_0^2 = F_1^2 = F_2^2 = F_3^2 = F_4^2 = 1$, $F_A F_B = -F_B F_A$ ($A \neq B$; $A, B = 0, 1, 2, 3, 4$). Define the pseudoscalar

$$\mathbf{i} = F_0 F_1 F_2 F_3 F_4, \qquad \mathbf{i}^2 = -1, \qquad \mathbf{i} F_A = F_A \mathbf{i}, \qquad A = 0, 1, 2, 3, 4 \tag{24}$$

Define

$$\mathcal{E}_\mu = F_\mu F_4 \tag{25}$$

We can immediately verify that $\mathcal{E}_\mu \mathcal{E}_\nu + \mathcal{E}_\nu \mathcal{E}_\mu = 2\eta_{\mu\nu}$. Taking into account that $\mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{4,1}^+$, we can explicitly exhibit here this isomorphism by considering the map $g: \mathcal{C}\ell_{1,3} \to \mathcal{C}\ell_{4,1}^+$ generated by the linear extension of the map $g^\#: \mathbb{R}^{1,3} \to \mathcal{C}\ell_{4,1}^+$, $g^\#(E_\mu) = \mathcal{E}_\mu = F_\mu F_4$, where E_μ ($\mu = 0, 1, 2, 3$) is an orthogonal basis of $\mathbb{R}^{1,3}$. Also $g(1_{\mathcal{C}\ell_{1,3}}) = 1_{\mathcal{C}\ell_{4,1}^+}$, where $1_{\mathcal{C}\ell_{1,3}}$ and $1_{\mathcal{C}\ell_{4,1}^+}$ are the identity elements in $\mathcal{C}\ell_{1,3}$ and $\mathcal{C}\ell_{4,1}^+$. Now consider the primitive idempotent of $\mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{4,1}^+$,

$$e_{41} = g(e) = \tfrac{1}{2}(1 + \mathcal{E}_0) \tag{26}$$

and the minimal left ideal $I_{4,1}^+ = \mathcal{C}\ell_{4,1}^+ e_{4,1}$. The elements $Z_{\Sigma_0} \in I_{4,1}^+$ can be written in an analogous way to $\Phi \in \mathcal{C}\ell_{1,3}\frac{1}{2}(1 + E_0)$ [equation (22)], i.e.,

$$Z_{\Sigma_0} = \sum z_i \bar{s}_i \tag{27}$$

where

$$\bar{s}_1 = e_{41}, \qquad \bar{s}_2 = -\mathcal{E}_1 \mathcal{E}_3 e_{41}, \qquad \bar{s}_3 = \mathcal{E}_3 \mathcal{E}_0 e_{41}, \qquad \bar{s}_4 = \mathcal{E}_1 \mathcal{E}_0 e_{41} \tag{28}$$

and

$$z_i = a_i + \mathscr{E}_2\mathscr{E}_1 b_i$$

and formally complex numbers, a_i, $b_i \in \mathbb{R}$.

Consider now the element $f_{\Sigma_0} \in \mathscr{C}\ell_{4,1}$,

$$\begin{aligned} f_{\Sigma_0} &= e_{41}\tfrac{1}{2}(1 + \mathbf{i}\mathscr{E}_1\mathscr{E}_2) \\ &= \tfrac{1}{2}(1 + \mathscr{E}_0)\tfrac{1}{2}(1 + \mathbf{i}\mathscr{E}_1\mathscr{E}_2) \end{aligned} \tag{29}$$

with $\mathbf{i}$ given by equation (24).

Since $f_{\Sigma_0}\mathscr{C}\ell_{4,1}f_{\Sigma_0} = \mathbb{C}f_{\Sigma_0} = f_{\Sigma_0}\mathbb{C}$ it follows that f_{Σ_0} is a primitive idempotent of $\mathscr{C}\ell_{4,1}$. We can easily show that each $\Phi_{\Sigma_0} \in I_{\Sigma_0} = \mathscr{C}\ell_{4,1}f_{\Sigma_0}$ can be written

$$\Psi_{\Sigma_0} = \sum_i \psi_i f_i, \qquad \psi_i \in \mathbb{C}$$

$$f_1 = f_{\Sigma_0}, \qquad f_2 = -\mathscr{E}_1\mathscr{E}_3 f_{\Sigma_0}, \qquad f_3 = \mathscr{E}_3\mathscr{E}_0 f_{\Sigma_0}, \qquad f_4 = \mathscr{E}_1\mathscr{E}_0 f_{\Sigma_0} \tag{30}$$

With the methods described in Vaz and Rodrigues (1993a) and Figueiredo *et al.* (1990a) we find the following representation in $M_4(\mathbb{C})$ for the generators $\mathscr{E}_\mu$ of $\mathscr{C}\ell_{4,1}^+ \simeq \mathscr{C}\ell_{1,3}$:

$$\mathscr{E}_0 \mapsto \underline{\gamma}_0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix} \leftrightarrow \mathscr{E}_i \mapsto \underline{\gamma}_i = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \tag{31}$$

where 1_2 is the unit 2×2 matrix and σ_i ($i = 1, 2, 3$) are the standard Pauli matrices. We immediately recognize the $\underline{\gamma}$-matrices in (31) as the standard ones appearing, e.g., in Bjorken and Drell (1964).

The matrix representation of $\Psi_{\Sigma_0} \in I_{\Sigma_0}$ will be denoted by the same letter without the index, i.e., $\Psi_{\Sigma_0} \mapsto \Psi \in M_4(\mathbb{C})f$, where

$$f = \tfrac{1}{2}(1 + \underline{\gamma}_0)\tfrac{1}{2}(1 + i\underline{\gamma}_1\underline{\gamma}_2), \qquad i = \sqrt{-1} \tag{32}$$

We have

$$\Psi = \begin{pmatrix} \psi_1 & 0 & 0 & 0 \\ \psi_2 & 0 & 0 & 0 \\ \psi_3 & 0 & 0 & 0 \\ \psi_4 & 0 & 0 & 0 \end{pmatrix}, \qquad \psi_i \in \mathbb{C} \tag{33}$$

Equations (22), (27), and (30) are enough to prove that there are bijections between the elements of the ideals $\mathscr{C}\ell_{3,1}\tfrac{1}{2}(1 + E_0)$, $\mathscr{C}\ell_{4,1}^+\tfrac{1}{2}(1 + \mathscr{E}_0)$, and $\mathscr{C}\ell_{4,1}$ $\tfrac{1}{2}(1 + \mathscr{E}_0)\tfrac{1}{2}(1 + \mathbf{i}\mathscr{E}_1\mathscr{E}_2)$.

We can easily find that the following relation exists between $\Psi_{\Sigma_0} \in \mathscr{C}\ell_{4,1}f_{\Sigma_0}$ and $Z_{\Sigma_0} \in \mathscr{C}\ell_{4,1}^+\tfrac{1}{2}(1 + \mathscr{E}_0)$:

$$\Psi_{\Sigma_0} = Z_{\Sigma_0}\tfrac{1}{2}(1 + \mathbf{i}\mathscr{E}_1\mathscr{E}_2) \tag{34}$$

Decomposing Z_{Σ_0} into even and odd parts relative to the $\mathbf{Z}_2$-graduation of $\mathscr{C}\ell^+_{4,1} \simeq \mathscr{C}\ell_{1,3}$, $Z_{\Sigma_0} = Z^+_{\Sigma_0} + Z^-_{\Sigma_0}$, we obtain $Z^+_{\Sigma_0} = Z^-_{\Sigma_0}\mathscr{E}_0$, which clearly shows that all information of Z_{Σ_0} is contained in $Z^+_{\Sigma_0}$. Then,

$$\Psi_{\Sigma_0} = Z^+_{\Sigma_0}\tfrac{1}{2}(1 + \mathscr{E}_0)\tfrac{1}{2}(1 + \mathbf{i}\mathscr{E}_1\mathscr{E}_2) \tag{35}$$

Now, if we take into account (Figueiredo *et al.*, 1990a) that $\mathscr{C}\ell^{++}_{4,1}\tfrac{1}{2}(1 + \mathscr{E}_0) = \mathscr{C}\ell^+_{4,1}\tfrac{1}{2}(1 + \mathscr{E}_0)$, where the symbol $\mathscr{C}\ell^{++}_{4,1}$ means $\mathscr{C}\ell^{++}_{4,1} \simeq \mathscr{C}\ell^+_{1,3} \simeq \mathscr{C}\ell_{3,0}$, we see that each $Z_{\Sigma_0} \in \mathscr{C}\ell^+_{4,1}\tfrac{1}{2}(1 + \mathscr{E}_0)$ can be written

$$Z^-_{\Sigma_0} = \psi_{\Sigma_0}\tfrac{1}{2}(1 + \mathscr{E}_0), \qquad \psi_{\Sigma_0} \in (\mathscr{C}\ell^+_{4,1})^+ \simeq \mathscr{C}\ell^+_{1,3} \tag{36}$$

Then putting $Z^+_{\Sigma_0} = \psi_{\Sigma_0}/2$, we can write equation (35) as

$$\begin{aligned}\Psi_{\Sigma_0} &= \psi\tfrac{1}{2}(1 + \mathscr{E}_0)\tfrac{1}{2}(1 + \mathbf{i}\mathscr{E}_1\mathscr{E}_2)\\ &= Z_{\Sigma_0}\tfrac{1}{2}(1 + \mathbf{i}\mathscr{E}_1\mathscr{E}_2)\end{aligned} \tag{37}$$

The matrix representations of Z_{Σ_0} and ψ_{Σ_0} in $M_4(\mathbb{C})$ (denoted by the same letter without index) in the spinorial basis given by (30) are

$$\Psi = \begin{pmatrix} \psi_1 & -\psi_2^* & \psi_3 & \psi_4^* \\ \psi_2 & \psi_1^* & \psi_4 & -\psi_3^* \\ \psi_3 & \psi_4^* & \psi_1 & -\psi_2^* \\ \psi_4 & -\psi_3^* & \psi_2 & \psi_1^* \end{pmatrix}, \qquad Z = \begin{pmatrix} \psi_1 & -\psi_2^* & 0 & 0 \\ \psi_2 & \psi_1^* & 0 & 0 \\ \psi_3 & \psi_4^* & 0 & 0 \\ \psi_4 & -\psi_3^* & 0 & 0 \end{pmatrix} \tag{38}$$

2.5. Algebraic Spinors for $\mathbb{R}^{p,q}$

Let $\mathscr{B}_\Sigma = \{\Sigma_0, \dot{\Sigma}, \ddot{\Sigma}, \ldots\}$ be the set of all ordered orthonormal bases for $\mathbb{R}^{p,q}$, i.e., each $\Sigma \in \mathscr{B}_\Sigma$ is the set $\Sigma = \{E_1, \ldots, E_p, E_{p+1}, \ldots, E_{p+q}\}$, $E_1^2 = \cdots = E_p^2 = 1$, $E_{p+1}^2 = \cdots = E_{p+q}^2 = -1$, $E_rE_s = -E_sE_r$ ($r \neq s$; $r, s = 1, 2, \ldots, p + q = n$). Any two bases, say, $\Sigma_0, \dot{\Sigma} \in \mathscr{B}_\Sigma$, are related by an element of the group $\mathbf{Spin}_+(p, q) \subset \Gamma_{pq}$. We write

$$\dot{\Sigma} = u\Sigma_0u^{-1}, \qquad u \in \mathbf{Spin}_+(p, q) \tag{39}$$

A primitive idempotent determined in a given basis $\Sigma \in \mathscr{B}_\Sigma$ will be denoted e_Σ. Then the idempotents e_{Σ_0}, $e_{\dot{\Sigma}}$, $e_{\ddot{\Sigma}}$, etc., such that, e.g.,

$$e_{\dot{\Sigma}} = ue_{\Sigma_0}u^{-1}, \qquad u \in \mathbf{Spin}_+(p, q) \tag{40}$$

define ideals I_{Σ_0}, $I_{\dot{\Sigma}}$, $I_{\ddot{\Sigma}}$, etc., that are geometrically equivalent according to the definition given by (19). We have

$$I_{\dot{\Sigma}} = uI_{\Sigma_0}u^{-1}, \qquad u \in \mathbf{Spin}_+(p, q) \tag{41}$$

but since $uI_{\Sigma_0} \equiv I_{\Sigma_0}$, equation (41) can also be written

$$I_{\dot{\Sigma}} = I_{\Sigma_0} u^{-1} \tag{42}$$

Equation (42) defines a new correspondence for the elements of the ideals, I_{Σ_0}, $I_{\dot{\Sigma}}$, $I_{\ddot{\Sigma}}$, etc. This suggests the following.

Definition. An algebraic spinor for $\mathbb{R}^{p,q}$ is an equivalence class of the quotient set $\{I_\Sigma\}/R$, where $\{I_\Sigma\}$ is the set of all geometrically equivalent ideals, and $\Psi_{\Sigma_0} \in I_{\Sigma_0}$ and $\Psi_{\dot{\Sigma}} \in I_{\dot{\Sigma}}$ are equivalent, $\Psi_{\dot{\Sigma}} \simeq \Psi_{\Sigma_0}$ (mod R) if and only if

$$\Psi_{\dot{\Sigma}} = \Psi_{\Sigma_0} u^{-1} \tag{43}$$

Ψ_Σ will be called the representative of the algebraic spinor in the basis $\Sigma \in \mathcal{B}_\Sigma$. Recall that $\dot{\Sigma} = u\Sigma u^{-1} = L\Sigma$, $u \in \mathbf{Spin}_+(1, 3)$, $L \in \mathcal{L}_+^\uparrow$.

2.6. What Is a Covariant Dirac Spinor (CDS)?

As we already know, $f_{\Sigma_0} = \frac{1}{2}(1 + \mathscr{E}_0)(1 + \mathbf{i}\mathscr{E}_1\mathscr{E}_2)$ [equation (29)] is a primitive idempotent of $\mathcal{C}\ell_{4,1} \simeq M_4(\mathbb{C})$. If $u \in \mathbf{Spin}_+(1, 3) \subseteq \mathbf{Spin}_+(4, 1)$, then all ideals $I_{\dot{\Sigma}} = I_{\Sigma_0}u^{-1}$ are geometrically equivalent to I_{Σ_0}. Since $\Sigma_0 = \{\mathscr{E}_0, \mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3\}$ is a basis for $\mathbb{R}^{1,3} \subset \mathcal{C}\ell_{4,1}^+$, the meaning of $\dot{\Sigma} = u\Sigma_0 u^{-1}$ is clear. From (30) we can write

$$I_{\Sigma_0} \ni \Psi_{\Sigma_0} = \sum \psi_i f_i \qquad \text{and} \qquad I_{\dot{\Sigma}} \ni \Psi_{\dot{\Sigma}} = \sum \dot{\psi}_i \dot{f}_i \tag{44}$$

where

$$f_1 = f_{\Sigma_0}, \qquad f_2 = -\mathscr{E}_1\mathscr{E}_3 f_{\Sigma_0}, \qquad f_3 = \mathscr{E}_3\mathscr{E}_0 f_{\Sigma_0}, \qquad f_4 = \mathscr{E}_1\mathscr{E}_0 f_{\Sigma_0}$$

and

$$\dot{f}_1 = f_{\dot{\Sigma}}, \qquad \dot{f}_2 = -\overline{\mathscr{E}}_1\overline{\mathscr{E}}_3 f_{\dot{\Sigma}}, \qquad \dot{f}_3 = \overline{\mathscr{E}}_3\overline{\mathscr{E}}_0 f_{\dot{\Sigma}}, \qquad \dot{f}_4 = \overline{\mathscr{E}}_1\overline{\mathscr{E}}_0 f_{\dot{\Sigma}}$$

Since $\Psi_{\dot{\Sigma}} = \Psi_{\Sigma_0}u^{-1}$, we get

$$\Psi_{\dot{\Sigma}} = \sum_i \psi_i u^{-1} \dot{f}_i = \sum_{i,k} S_{ik}(u^{-1})\psi_i \dot{f}_k = \sum_k \dot{\psi}_k \dot{f}_k$$

Then

$$\dot{\psi}_k = \sum_i S_{ik}(u^{-1})\psi_i \tag{45}$$

where $S_{ik}(u^{-1})$ are the matrix components of the representation in $M_4(\mathbb{C})$ of $u^{-1} \in \mathbf{Spin}_+(1, 3)$. As proved in Vaz and Rodrigues (1993a) and Figueiredo *et al.* (1990a), the matrices $S(u)$ correspond to the representation $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $SL(2, \mathbb{C}) \simeq \mathbf{Spin}_+(1, 3)$.

We remark that all the elements of the set $\{I_{\dot{\Sigma}}\}$ of the ideals geometrically equivalent to I_{Σ_0} under the action of $u \in \mathbf{Spin}_+(1, 3) \subset \mathbf{Spin}_+(4, 1)$ have the same image $I = M_4(\mathrm{C})f$, where f is given by (32), i.e.,

$$f = \tfrac{1}{2}(1 + \underline{\gamma}_0)(1 + i\underline{\gamma}_1\underline{\gamma}_2), \qquad i = \sqrt{-1}$$

where $\underline{\gamma}_\mu$, $\mu = 0, 1, 2, 3$ are the Dirac matrices given by (31).

Then, if

$$\begin{aligned} \gamma\colon \quad \mathscr{C}\ell_{4,1} &\to M_4(\mathrm{C}) \equiv \mathrm{End}(M_4(\mathrm{C})f) \\ x &\mapsto \gamma(x)\colon \quad M_4(\mathrm{C})f \to M_4(\mathrm{C})f \end{aligned} \tag{46}$$

it follows that $\gamma(\mathscr{E}_\mu) = \gamma(\dot{\mathscr{E}}_\mu) = \underline{\gamma}_\mu$, $\gamma(f_{\Sigma_0}) = \gamma(f_{\dot{\Sigma}}) = f$ for all $\mathscr{E}_\mu$, $\dot{\mathscr{E}}_\mu$ such that $\dot{\mathscr{E}}_\mu = u\mathscr{E}_\mu u^{-1}$ for some $u \in \mathbf{Spin}_+(1, 3)$. Observe that all the information concerning the orthonormal frames Σ_0, $\dot{\Sigma}$, etc., disappear in the matrix representation of the ideals I_{Σ_0}, $I_{\dot{\Sigma}}$, . . . in $M_4(\mathrm{C})$, since all these ideals are mapped in the same ideal $I = M_4(\mathrm{C})f$.

With the above remark and taking into account equation (45), we are then led to the following.

Definition. A covariant Dirac spinor (CDS) for $\mathrm{R}^{1,3}$ is an equivalent class of triplets $(\Sigma, S(u), \Psi)$, Σ being an orthonormal basis of $\mathrm{R}^{1,3}$, $S(u) \in D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathbf{Spin}_+(1, 3)$, $u \in \mathbf{Spin}_+(1, 3)$, and $\Psi \in M_4(\mathrm{C})f$; and

$$(\Sigma, S(u), \Psi) \sim (\Sigma_0, S(u_0), \Psi_0)$$

if and only if

$$\begin{aligned} \Psi &= S(u)S^{-1}(u_0)\Psi_0, \qquad \mathscr{H}(uu_0^{-1}) \\ &= L\Sigma_0, \qquad L \in \mathscr{L}_+^\uparrow, \qquad u \in \mathbf{Spin}_+(1, 3) \end{aligned} \tag{47}$$

The pair $(\Sigma, S(u))$ is called a spinorial frame. Observe that the CDS just defined depends on the choice of the original spinorial frame (Σ_0, u_0) and, obviously, to different possible choices there correspond isomorphic ideals in $M_4(\mathrm{C})$. For simplicity we can fix $u_0 = 1$, $S(u_0) = 1$.

The definition of CDS just given agrees with that given by Choquet-Bruhat (1968) except for the irrelevant fact that Choquet-Bruhat uses as the space of representatives of a CDS the complex four-dimensional vector space C^4 instead of $I = M_4(\mathrm{C})f$. We see that Choquet-Bruhat's definition is well justified from the point of view of the theory of algebraic spinors presented above.

2.7. Algebraic Dirac Spinors (ADS) and Dirac–Hestenes Spinors (DHS)

We saw in Section 2.4 that there is bijection between $\psi_{\Sigma_0} \in \mathcal{C}\ell_{4,1}^{++} \simeq \mathcal{C}\ell_{1,3}^{+}$ and $\Psi_{\Sigma_0} \in I_{\Sigma_0} = \mathcal{C}\ell_{4,1}^{+} f_{\Sigma_0}$, namely [equation (37)]

$$\Psi_{\Sigma_0} = \psi_{\Sigma_0}\tfrac{1}{2}(1 + \mathscr{E}_0)\tfrac{1}{2}(1 + \mathbf{i}\mathscr{E}_1\mathscr{E}_2)$$

Then, as we already said, all information contained in Ψ_{Σ_0} (which is the representative in the basis Σ_0 of an algebraic spinor for $\mathbb{R}^{1,3}$) is also contained in $\psi_{\Sigma_0} \in \mathcal{C}\ell_{4,1}^{++} \simeq \mathcal{C}\ell_{1,3}^{+}$. We are then lead to the following.

Definition. Consider the quotient set $\{I_\Sigma\}/\mathfrak{R}$, where $\{I_\Sigma\}$ is the set of all geometrically equivalent minimal left ideals of $\mathcal{C}\ell_{1,3}$ generated by $e_{\Sigma_0} = \frac{1}{2}(1 + E_0)$, $\Sigma_0 = (E_0, E_1, E_2, E_3)$ [i.e., $I_{\dot\Sigma}$, $I_{\ddot\Sigma} \in \{I_\Sigma\}$, then $I_{\ddot\Sigma} = uI_{\dot\Sigma}u^{-1} \equiv I_{\dot\Sigma}u^{-1}$ for some $u \in \mathbf{Spin}_+(1, 3)$]. An algebraic Dirac spinor (ADS) is an element of $\{I_\Sigma\}/\mathfrak{R}$. Then, if $\Phi_{\dot\Sigma} \in I_{\dot\Sigma}$, $\Phi_{\ddot\Sigma} \in I_{\ddot\Sigma}$, then $\Phi_{\ddot\Sigma} \simeq \Phi_{\dot\Sigma}(\text{mod } \mathfrak{R})$ if and only if $\Phi_{\ddot\Sigma} = \Phi_{\dot\Sigma}u^{-1}$, for some $u \in \mathbf{Spin}_+(1, 3)$.

We remark that [see equation (36)]

$$\Phi_{\dot\Sigma} = \psi_{\dot\Sigma}e_{\dot\Sigma}, \qquad \Phi_{\ddot\Sigma} = \psi_{\ddot\Sigma}e_{\ddot\Sigma}, \qquad \psi_{\dot\Sigma}, \psi_{\ddot\Sigma} \in \mathcal{C}\ell_{1,3}^{+}$$

and since $e_{\ddot\Sigma} = ue_{\dot\Sigma}u^{-1}$ for some $u \in \mathbf{Spin}_+(1, 3)$ we get[5]

$$\psi_{\ddot\Sigma} = \psi_{\dot\Sigma}u^{-1} \tag{48}$$

Now, we quoted in Section 2.3 that for $p + q \leq 5$, $\mathbf{Spin}_+(p, q) = \{u \in \mathcal{C}\ell_{p,q}^{+} \mid u\tilde{u} = 1\}$. Then for all $\psi_\Sigma \in \mathcal{C}\ell_{1,3}^{+}$ such that $\psi_\Sigma\tilde{\psi}_\Sigma \neq 0$ we obtain immediately the polar form

$$\psi_\Sigma = \rho^{1/2}e^{\beta E_5/2}R_\Sigma \tag{49}$$

where $\rho \in \mathbb{R}^+$, $\beta \in \mathbb{R}$, $R_\Sigma \in \mathbf{Spin}_+(1, 3)$, $E_5 = E_0E_1E_2E_3$. With the above remark in mind we present the following.

Definition. A Dirac–Hestenes spinor (DHS) is an equivalence class of triplets (Σ, u, ψ_Σ), where Σ is an oriented orthonormal basis of $\mathbb{R}^{1,3} \subset \mathcal{C}\ell_{1,3}$, $u \in \mathbf{Spin}_+(1, 3)$, and $\psi_\Sigma \in \mathcal{C}\ell_{1,3}^{+}$). We say that $(\Sigma, u, \psi_\Sigma) \sim (\Sigma_0, u_0, \psi_{\Sigma_0})$ if and only if $\psi_\Sigma = \psi_{\Sigma_0}u_0^{-1}u$, $\mathscr{H}(uu_0^{-1}) = L$, $\Sigma = L\Sigma_0$ $(\equiv u^{-1}u_0\Sigma_0u_0^{-1}u)$, u, $u_0 \in \mathbf{Spin}_+(1, 3)$, $L \in \mathscr{L}_+^{\uparrow}$. Here u_0 is arbitrary but fixed. A DHS determines a set of vectors $X_\mu \in \mathbb{R}^{1,3}$ $(\mu = 0, 1, 2, 3)$ by a given representative $\psi_{\dot\Sigma}$ of the DHS in the basis Σ by

$$\psi\colon\ \dot\Sigma \to \mathbb{R}^{1,3}, \qquad \psi_{\dot\Sigma}\dot{E}_\mu\tilde{\psi}_{\dot\Sigma} = X_\mu, \qquad \dot\Sigma = (\dot{E}_0, \dot{E}_1, \dot{E}_2, \dot{E}_3) \tag{50}$$

We give yet another equivalent definition of a DHS:

[5] Lounesto (1993, 1994) calls 2Φ the mother of all the real spinors.

Definition. A Dirac–Hestenes spinor is an element of the quotient set $\mathcal{C}\ell^+_{1,3}/\mathcal{R}$ such that given the basis Σ, $\dot{\Sigma}$ of $\mathbb{R}^{1,3} \subset \mathcal{C}\ell_{1,3}$, $\psi_\Sigma \in \mathcal{C}\ell^+_{1,3}$, $\psi_{\dot{\Sigma}} \in \mathcal{C}\ell^+_{1,3}$, then $\psi_{\dot{\Sigma}} \sim \psi_\Sigma (\mathrm{mod}\, \mathcal{R})$ if and only if $\psi_{\dot{\Sigma}} = \psi_\Sigma u^{-1}$, $\dot{\Sigma} = L\Sigma = u\Sigma u^{-1}$, $\mathcal{H}(u) = L$, $u \in \mathbf{Spin}_+(1, 3)$, $L \in \mathcal{L}^\uparrow_+$.

With the canonical form of a DHS given by equation (49) some features of the hidden geometrical nature of the Dirac spinors defined above comes to light: equation (49) says that when $\psi_\Sigma \tilde{\psi}_\Sigma \neq 0$ the Dirac–Hestenes spinor ψ_Σ is equivalent to a Lorentz rotation followed by a dilation and a duality mixing given by the term $\exp(\beta E_5/2)$, where β is the so-called Yvon–Takabayasi angle (Yvon, 1940; Takabayasi, 1957) and the justification for the name duality rotation can be found in Vaz and Rodrigues (1993a). We emphasize that the definition of the Dirac–Hestenes spinors given above is new. In the past objects $\psi \in \mathcal{C}\ell^+_{1,3}$ satisfying $\psi X \tilde{\psi} = Y$ for $X, Y \in \mathbb{R}^{1,3} \subset \mathcal{C}\ell_{1,3}$ have been called operator spinors (see, e.g., Hestenes and Sobczyk (1984), Lounesto (1993a,b). DHS have been used as the departure point of many interesting results (e.g., Vaz and Rodrigues, 1993a, 1994; Pavšič *et al.*, 1993; Rodrigues *et al.*, 1993a).

2.8. Fierz Identities

The formulation of the Fierz (1937) identities using the CDS $\Psi \in \mathbb{C}^4$ is well known (Crawford, 1985). Here we present the identities for $\Psi_{\Sigma_0} \in I_{\Sigma_0} \simeq (\mathbb{C} \otimes \mathcal{C}\ell_{1,3}) f_{\Sigma_0}$ and for the DHS $\psi_{\Sigma_0} \in \mathcal{C}\ell^+_{1,3}$ (Lounesto, 1993, 1994). Let then $\Psi \in \mathbb{C}^4$ be a representative of a CDS for $\mathbb{R}^{1,3}$ associated to the basis $\Sigma_0 = \{E_0, E_1, E_2, E_3\}$ of $\mathbb{R}^{1,3} \subset \mathcal{C}\ell_{1,3}$. Then Ψ, Ψ_{Σ_0} determine the following so-called bilinear covariants:

$$
\begin{aligned}
\sigma &= \Psi^\dagger \gamma_0 \Psi = 4\langle \tilde{\Psi}^*_{\Sigma_0} \Psi_{\Sigma_0} \rangle_0 \\
J_\mu &= \Psi^\dagger \gamma_0 \gamma_\mu \Psi = 4\langle \tilde{\Psi}^*_{\Sigma_0} E_\mu \Psi_{\Sigma_0} \rangle_0 \\
S_{\mu\nu} &= \Psi^\dagger \gamma_0 i \gamma_{\mu\nu} \Psi = 4\langle \tilde{\Psi}^*_{\Sigma_0} i E_{\mu\nu} \Psi_{\Sigma_0} \rangle_0 \\
K_\mu &= \Psi^\dagger \gamma_0 i \gamma_{0123} \Psi = 4\langle \tilde{\Psi}^*_{\Sigma_0} i E_{0123} E_\mu \Psi_{\Sigma_0} \rangle_0 \\
\omega &= -\Psi^\dagger \gamma_0 \gamma_{0123} \Psi = -4\langle \tilde{\Psi}^*_{\Sigma_0} E_{0123} \Psi_{\Sigma_0} \rangle_0
\end{aligned}
\tag{51}
$$

where † means Hermitian conjugation and * complex conjugation. We remark that the reversion in $\mathcal{C}\ell_{4,1}$ corresponds to the reversion plus complex conjugation in $\mathbb{C} \otimes \mathcal{C}\ell_{1,3}$.

All the bilinear covariants are real and have physical meaning in the Dirac theory of the electron, but its geometrical nature appears clearly when these bilinear covariants are formulated with the aid of the DHS.

Introducing the Hodge dual of a Clifford number $X \in \mathscr{C}\ell_{1,3}$ by

$$\star X = \tilde{X}E_5, \qquad E_5 = E_0E_1E_2E_3 \tag{52}$$

we find that the bilinear covariants given by (51) become, in terms of ψ_{Σ_0}, the representative of a DHS in the orthonormal basis $\Sigma_0 = \{E_0, E_1, E_2, E_3\}$ of $\mathbb{R}^{1,3} \subset \mathscr{C}\ell_{1,3}$,

$$\begin{aligned}
\psi_{\Sigma_0}\tilde{\psi}_{\Sigma_0} &= \sigma + \star\omega, & J &= J_\mu E^\mu \\
\psi_{\Sigma_0}E_0\tilde{\psi}_{\Sigma_0} &= J, & S &= \tfrac{1}{2}S_{\mu\nu}E^\mu E^\nu \\
\psi_{\Sigma_0}E_1E_2\tilde{\psi}_{\Sigma} &= S, & K &= K_\mu E^\mu \\
\psi_{\Sigma_0}E_3\tilde{\psi}_{\Sigma_0} &= K, & E^\mu &= \eta^{\mu\nu}E_\mu \\
\psi_{\Sigma_0}E_0E_3\tilde{\psi}_{\Sigma_0} &= \star S, & \eta^{\mu\nu} &= \mathrm{diag}(1, -1, -1, -1) \\
\psi_{\Sigma_0}E_0E_1E_2\tilde{\psi}_{\Sigma_0} &= \star K
\end{aligned} \tag{53}$$

The Fierz identities are

$$J^2 = \sigma^2 + \omega^2, \qquad J\cdot K = 0, \qquad J^2 = -K^2, \qquad J \wedge K = -(\omega + \star\sigma)S \tag{54}$$

$$\begin{cases} S\cdot J = \omega K, & S\cdot K = \omega J \\ (\star S)\cdot J = -\sigma K, & (\star S)\cdot K = -\sigma J \\ S\cdot S = \omega^2 - \sigma^2 & (\star S)\cdot S = -2\sigma\omega \end{cases} \tag{55}$$

$$\begin{cases} JS = -(\omega + \star\sigma)K, & KS = -(\omega + \star\sigma)J \\ SJ = -(\omega - \star\sigma)K, & SK = -(\omega - \star\sigma)J \\ S^2 = \omega^2 - \sigma^2 - 2\sigma(\star\omega) \\ S^{-1} = -S(\sigma - \star\omega)^2/(\sigma^2 + \omega^2) = KSK/(\sigma^2 + \omega^2)^2 \end{cases} \tag{56}$$

The proof of these identities using the DHS is almost a triviality.

The importance of the bilinear covariants is due to the fact that we can recover from them the CDS $\Psi_{\Sigma_0} \in M_4(\mathbb{C})f$ or all other kinds of Dirac spinors defined above through an algorithm due to Crawford (see also Lounesto, 1993, 1994). Indeed, representing the images of the bilinear covariants in $\mathscr{C}\ell_{1,3}$ and $\mathscr{C}\ell^+_{4,1} \subset \mathscr{C}\ell_{4,1}$ under the mapping g [equation (25)] by the same letter, we have that the following result holds true: let

$$Z_{\Sigma_0} = (\sigma + J + iS + i(\star K) + \star\omega) \in \mathbb{C} \otimes \mathscr{C}\ell_{1,3} \tag{57}$$

where σ, J, S, K, ω are the bilinear covariants of $\Psi_{\Sigma_0} \simeq (\mathbb{C} \otimes \mathscr{C}\ell_{1,3})f_{\Sigma_0}$. Take $\eta_{\Sigma_0} \in (\mathbb{C} \otimes \mathscr{C}\ell_{1,3})f_{\Sigma_0}$ such that $\tilde{\eta}^*_{\Sigma_0}\Psi_{\Sigma_0} \neq 0$. Then Ψ_{Σ_0} and $Z_{\Sigma_0}\eta_{\Sigma_0}$ differ by a complex factor. We have

$$\Psi_{\Sigma_0} = \frac{1}{4N_{\eta\Sigma_0}} e^{-i\alpha} Z_{\Sigma_0}\eta_{\Sigma_0} \tag{58}$$

$$N_{\eta\Sigma_0} = (\langle\tilde{\eta}^*_{\Sigma_0} Z_{\Sigma_0}\eta_{\Sigma_0}\rangle_0)^{1/2}, \qquad e^{-i\alpha} = \frac{4}{N\eta_{\Sigma_0}}\langle\tilde{\eta}^*\Psi\rangle_0 \tag{59}$$

Choosing $\eta_{\Sigma_0} = f_{\Sigma_0}$, we obtain

$$N_{f\Sigma_0} = \tfrac{1}{2}[\sigma + J\cdot E_0 - S\cdot(E_1E_2) - K\cdot E_3]^{1/2}, \qquad e^{-i\alpha} = \psi_1/|\psi_1| \tag{60}$$

where ψ_1 is the first component of Ψ_{Σ_0} in the spinorial basis $\{s_i\}$.

It is easier to recuperate the CDS from its bilinear covariants if we use the DHS $\psi_{\Sigma_0} \in \mathscr{C}\ell^+_{1,3} \simeq (\mathscr{C}\ell^+_{4,1})^+$, since putting

$$\begin{cases}\psi_{\Sigma_0}(1 + E_0)\tilde{\psi}_{\Sigma_0} = P\\ \psi_{\Sigma_0}(1 + E_0)E_1E_2\tilde{\psi}_{\Sigma_0} = Q\end{cases} \tag{61}$$

$$\psi_{\Sigma_0}(1 + E_0)(1 + \mathbf{i}E_1E_2)\tilde{\psi}_{\Sigma_0} = (P + \mathbf{i}Q) \tag{62}$$

results in

$$P = \sigma + J + \omega, \qquad Q = S + \star K \tag{63}$$

and

$$Z_{\Sigma_0} = P\frac{1}{2}\left(1 + \frac{\mathbf{i}}{2\sigma}Q\right)^2 \tag{64}$$

valid for $\sigma \neq 0$, $\omega \neq 0$ [for other cases see Lounesto (1994)]. From the above results it follows that Ψ_{Σ_0} can be easily determined from its bilinear covariant except for a "complex" E_2E_1 phase factor.

3. THE CLIFFORD BUNDLE OF SPACETIME AND ITS IRREDUCIBLE MODULE REPRESENTATIONS

3.1. The Clifford Bundle of Spacetime

Let M be a four-dimensional, real, connected, paracompact manifold. Let TM [T^*M] be the tangent [cotangent] bundle of M.

Definition. A Lorentzian manifold is a pair (M, g), where $g \in \sec T^*M \times T^*M$ is a Lorentzian metric of signature (1, 3), i.e., for all $x \in M$, $T_xM \simeq T^*_xM \simeq \mathbb{R}^{1,3}$, where $\mathbb{R}^{1,3}$ is the vector Minkowski space.

Definition. A spacetime $\mathcal{M}$ is a triple (M, g, ∇), where (M, g) is a time-oriented and spacetime-oriented Lorentzian manifold and ∇ is a linear connection for M such that $\nabla g = 0$. If in addition $\mathbf{T}(\nabla) = 0$ and $\mathbf{R}(\nabla) \neq 0$, where $\mathbf{T}$ and $\mathbf{R}$ are respectively the torsion and curvature tensors, then $\mathcal{M}$ is said to be a Lorentzian spacetime. When $\nabla g = 0$, $\mathbf{T}(\nabla) = 0$, $\mathbf{R}(\nabla) = 0$, $\mathcal{M}$ is called Minkowski spacetime and will be denoted by $\mathbb{M}$. When $\nabla g = 0$, $\mathbf{T}(\nabla) \neq 0$ and $\mathbf{R}(\nabla) = 0$ or $\mathbf{R}(\nabla) \neq 0$, $\mathcal{M}$ is said to be a Riemann–Cartan spacetime.

In what follows $P_{SO_+(1,3)}(\mathcal{M})$ denotes the principal bundles of oriented Lorentz tetrads (Rodrigues and Figueiredo, 1990; Choquet-Bruhat *et al.*, 1982). By g^{-1} we denote the "metric" of the cotangent bundle.

It is well known that the natural operations on metric vector spaces, such as, e.g., direct sum, tensor product, exterior power, etc., carry over canonically to vector bundles with metrics. Take, e.g., the cotangent bundle T^*M. If π: $T^*M \to M$ is the canonical projection, then in each fiber $\pi^{-1}(x) = T_x^*M \simeq \mathbb{R}^{1,3}$ the "metric" g^{-1} can be used to construct a Clifford algebra $\mathcal{C}\ell(T_x^*M) \simeq \mathcal{C}\ell_{1,3}$. We have the following.

Definition. The Clifford bundle of spacetime $\mathcal{M}$ is the bundle of algebras

$$\mathcal{C}\ell(\mathcal{M}) = \bigcup_{x \in M} \mathcal{C}\ell(T_x^*M) \tag{65}$$

As is well known, $\mathcal{C}\ell(\mathcal{M})$ is the quotient bundle

$$\mathcal{C}\ell(\mathcal{M}) = \frac{\tau M}{\mathbf{J}(\mathcal{M})} \tag{66}$$

where $\tau M = \bigoplus_{r=0}^{\infty} T^{0,r}(M)$ and $T^{(0,r)}(M)$ is the space of r-covariant tensor fields, and $\mathbf{J}(\mathcal{M})$ is the bundle of ideals whose fibers at $x \in M$ are the two side ideals in τM generated by the elements of the form $a \otimes b + b \otimes a - 2g^{-1}(a, b)$ for $a, b \in T^*\mathcal{M}$.

Let π_c: $\mathcal{C}\ell(\mathcal{M}) \to M$ be the canonical projection of $\mathcal{C}\ell(\mathcal{M})$ and let $\{U_\alpha\}$ be an open covering of M. From the definition of a fiber bundle (Lichnerowicz, 1984) we know that there is a trivializing mapping $\varphi\alpha$: $\pi_c^{-1}(U_\alpha) \to U_\alpha \times \mathcal{C}\ell_{1,3}$ of the form $\varphi\alpha(p) = (\pi_c(p), \overset{\Delta}{\varphi}_\alpha(p))$. If $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and $x \in U_{\alpha\beta}$, $p \in \pi_c^{-1}(x)$, then

$$\overset{\Delta}{\varphi}_\alpha(p) = f_{\alpha\beta}(x)\overset{\Delta}{\varphi}_\beta(p) \tag{67}$$

for $f_{\alpha\beta}(x) \in \mathrm{Aut}(\mathcal{C}\ell_{1,3})$, where $f_{\alpha\beta}$: $U_{\alpha\beta} \to \mathrm{Aut}(\mathcal{C}\ell_{1,3})$ are the transition mappings of $\mathcal{C}\ell(\mathcal{M})$. We know that every automorphism of $\mathcal{C}\ell_{1,3}$ is inner and it follows that

$$f_{\alpha\beta}(x)\overset{\Delta}{\varphi}_\beta(p) = g_{\alpha\beta}(x)\overset{\Delta}{\varphi}_\beta(p)g_{\alpha\beta}(x)^{-1} \tag{68}$$

for some $g_{\alpha\beta}(x) \in \mathcal{C}\ell_{1,3}^\star$, the group of invertible elements of $\mathcal{C}\ell_{1,3}$. We can write equivalently instead of (68)

$$f_{\alpha\beta}(x)\overset{\Delta}{\varphi}_\beta(p) = \overset{\Delta}{\varphi}_\beta(a_{\alpha\beta}pa_{\alpha\beta}^{-1}) \tag{69}$$

for some invertible element $a_{\alpha\beta} \in \mathscr{C}\ell(T_x^* M)$.

Now, the group $SO_+(1, 3)$ has, as we know (Section 2), a natural extension in the Clifford algebra $\mathscr{C}\ell_{1,3}$. Indeed we know that $\mathscr{C}\ell_{1,3}^*$ acts naturally on $\mathscr{C}\ell_{1,3}$ as an algebra automorphism through its adjoint representation Ad: $u \mapsto \mathrm{Ad}_u$, $\mathrm{Ad}_u(a) = uau^{-1}$. Also, $\mathrm{Ad}|_{\mathbf{Spin}_+(1,3)} = \sigma$ defines a group homeomorphism σ: $\mathbf{Spin}_+(1, 3) \to SO_+(1, 3)$ which is onto with kernel $\mathbf{Z}_2$. It is clear, since Ad_{-1} = identity, that Ad: $\mathbf{Spin}_+(1, 3) \to \mathrm{Aut}(\mathscr{C}\ell_{1,3})$ descends to a representation of $SO_+(1, 3)$. Let us call Ad′ this representation, i.e., Ad': $SO_+(1, 3) \to \mathrm{Aut}(\mathscr{C}\ell_{1,3})$. Then we can write $\mathrm{Ad}'_{\sigma(u)}a = \mathrm{Ad}_u a = uau^{-1}$.

From this it is clear that the structure group of the Clifford bundle $\mathscr{C}\ell(\mathcal{M})$ is reducible from $\mathrm{Aut}(\mathscr{C}\ell_{1,3})$ to $SO_+(1, 3)$. This follows immediately from the existence of the Lorentzian structure (M, g) and the fact that $\mathscr{C}\ell(\mathcal{M})$ is the exterior bundle where the fibers are equipped with the Clifford product. Thus the transition maps of the principal bundle of oriented Lorentz tetrads $P_{SO_+(1,3)}(\mathcal{M})$ can be (through Ad′) taken as transition maps for the Clifford bundle. We then have the result (Blaine Lawson and Michelson, 1989)

$$\mathscr{C}\ell(\mathcal{M}) = P_{SO_+(1,3)}(\mathcal{M}) \times_{\mathrm{Ad}'} \mathscr{C}\ell_{1,3} \tag{70}$$

3.2. Spinor Bundles

Definition. A spinor structure for $\mathcal{M}$ consists of a principal fiber bundle π_s: $P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \to M$ with group $SL(2, \mathbb{C}) \simeq \mathbf{Spin}_+(1, 3)$ and a map

$$s\colon \quad P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \to P_{SO_+(1,3)}(\mathcal{M})$$

satisfying the following conditions:

1. $\pi(s(p)) = \pi_s(p)\ \forall p \in P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$
2. $s(pu) = s(p)\mathscr{H}(u)\ \forall p \in P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ and $\mathscr{H}$: $SL(2, \mathbb{C}) \to SO_+(1, 3)$.

Now, in Section 2 we learned that the minimal left (right) ideals of $\mathscr{C}\ell_{p,q}$ are irreducible left (right) module representations of $\mathscr{C}\ell_{p,q}$ and we defined covariant and algebraic Dirac spinors as elements of quotient sets of the type $\{I_\Sigma\}/R$ (Sections 2.6 and 2.7) in appropriate Clifford algebras. We defined also in Section 2 the DHS. We are now interested in defining algebraic Dirac spinor fields (ADSF) and also Dirac–Hestenes spinor fields (DHSF).

So, in the spirit of Section 2, the following question naturally arises: Is it possible to find a vector bundle π_s: $S(\mathcal{M}) \to M$ with the property that each fiber over $x \in M$ is an irreducible module over $\mathscr{C}\ell(T_x^* M)$?

The answer to the above question is in general no. Indeed it is well known (Milnor, 1963) that the necessary and sufficient condition for $S(\mathcal{M})$

to exist is that the spinor structure bundle $P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ exist, which implies the vanishing of the second Stiefel–Whitney class of M, i.e., $\omega_2(M) = 0$. For a spacetime $\mathcal{M}$ this is equivalent, as shown originally by Geroch (1968, 1970), to $P_{SO_+(1,3)}(\mathcal{M})$ being a trivial bundle, i.e., to its admitting a global section. When $P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ exists we say that $\mathcal{M}$ is a spin manifold.

Definition. A real spinor bundle for $\mathcal{M}$ is the vector bundle

$$S(\mathcal{M}) = P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_{\mu} \mathbf{M} \tag{71}$$

where $\mathbf{M}$ is a left (right) module for $\mathcal{C}\ell_{1,3}$ and where μ: $P_{\mathbf{Spin}_+(1,3)} \to SO_+(1, 3)$ is a representation given by left (right) multiplication by elements of $\mathbf{Spin}_+(1, 3)$.

Definition. A complex spinor bundle for $\mathcal{M}$ is the vector bundle

$$S_c(\mathcal{M}) = P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_{\mu_c} \mathbf{M}_c \tag{72}$$

where $\mathbf{M}$ is a complex left (right) module for $\mathrm{C} \otimes \mathcal{C}\ell_{1,3} \simeq \mathcal{C}\ell_{4,1} \simeq M_4(\mathrm{C})$, and where μ_c: $P_{\mathbf{Spin}_+(1,3)} \to SO_+(1, 3)$ is a representation given by left (right) multiplication by elements of $\mathbf{Spin}_+(1, 3)$.

Taking, e.g., $\mathbf{M}_c = \mathrm{C}^4$ and μ_c the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $\mathbf{Spin}_+(1, 3)$ in $\mathrm{End}(\mathrm{C}^4)$, we recognize immediately the usual definition of the covariant spinor bundle of $\mathcal{M}$, as given, e.g., in Choquet-Bruhat (1968).

Since, besides being right (left) linear spaces over $\mathbb{H}$, the left (right) ideals of $\mathcal{C}\ell_{1,3}$ are representation modules of $\mathcal{C}\ell_{1,3}$, we have the following.

Definition. $I(\mathcal{M})$ is a real spinor bundle for $\mathcal{M}$ such that $\mathbf{M}$ in equation (71) is I, a minimal left (right) ideal of $\mathcal{C}\ell_{1,3}$.

In what follows we fix the ideal taking $I = \mathcal{C}\ell_{1,3}\frac{1}{2}(1 + E_0) = \mathcal{C}\ell_{1,3}e$. If π_I: $I(\mathcal{M}) \to M$ is the canonical projection and $\{U_\alpha\}$ is an open covering of M, we know from the definition of a fiber bundle that there is a trivializing mapping $\chi_\alpha(q) = (\pi_I(q), \overset{\Delta}{\chi}_\alpha(q))$. If $U_{\alpha\beta} = U_\alpha \cap U_\beta$ and $x \in U_{\alpha\beta}$, $q \in \pi_I^{-1}(U_\alpha)$, then

$$\overset{\Delta}{\chi}_\alpha(q) = g_{\alpha\beta}(x)\overset{\Delta}{\chi}_\beta(q) \tag{73}$$

for the transition maps in $\mathbf{Spin}_+(1, 3)$.[6] Equivalently,

$$\overset{\Delta}{\chi}_\alpha(q) = \overset{\Delta}{\chi}_\beta(a_{\alpha\beta}q) \tag{74}$$

for some $a_{\alpha\beta} \in \mathcal{C}\ell(T_x^*\mathcal{M})$. Thus, for the transition maps to be in $\mathbf{Spin}_+(1, 3)$

[6] We start with transition maps in $\mathcal{C}\ell_{p,q}^*$ and then by the bundle reduction process we end with $\mathbf{Spin}_+(1, 3)$.

it is equivalent that the right action of $\mathbb{H}e = e\mathbb{H} = e\mathscr{C}\ell_{1,3}e$ be defined in the bundle, since for $q \in \pi_x^{-1}(x)$, $x \in U_\alpha$, and $a \in \mathbb{H}$ we define qa as the unique element of $\pi_q^{-1}(x)$ such that

$$\overset{\Delta}{\chi}_\alpha(qa) = \overset{\Delta}{\chi}_\alpha(q)a \tag{75}$$

Naturally, for the validity of (75) to make sense it is necessary that

$$g_{\alpha\beta}(x)(\overset{\Delta}{\chi}_\alpha(q)a) = (g_{\alpha\beta}(x)\overset{\Delta}{\chi}_\alpha(q))a \tag{76}$$

and (76) implies that the transition maps are $\mathbb{H}$-linear.[7]

Let $f_{\alpha\beta}$: $U_{\alpha\beta} \to \mathrm{Aut}(\mathscr{C}\ell_{1,3})$ be the transition functions for $\mathscr{C}\ell(\mathcal{M})$. On the intersection $U_\alpha \cap U_\beta \cap U_\alpha$ it must hold that

$$f_{\alpha\beta}f_{\beta\gamma} = f_{\alpha\gamma} \tag{77}$$

We say that a set of *lifts* of the transition functions of $\mathscr{C}\ell(\mathcal{M})$ is a set of elements in $\mathscr{C}\ell^\star_{1,3}$, $\{g_{\alpha\beta}\}$, such that if

$$\mathrm{Ad}\colon\ \mathscr{C}\ell^\star_{1,3} \to \mathrm{Aut}(\mathscr{C}\ell_{1,3})$$

$$\mathrm{Ad}(u)x = uXu^{-1}, \qquad \forall X \in \mathscr{C}\ell_{1,3}$$

then $\mathrm{Ad}_{g_{\alpha\beta}} = f_{\alpha\beta}$ in all intersections.

Using the theory of the Cěch cohomology (Benn and Tucker, 1988), it can be shown that any set of lifts can be used to define a characteristic class $\omega(\mathscr{C}\ell(\mathcal{M})) \in \check{H}^2(M, \mathbb{H}^*)$, the second Cěch cohomology group with values in $\mathbb{H}^*$, the space of all nonzero $\mathbb{H}$-valued germs of functions in M.

We say that we can coherently lift the transition maps $\mathscr{C}(\mathcal{M})$ to a set $\{g_{\alpha\beta}\} \in \mathscr{C}\ell^\star_{1,3}$ if in the intersection $U_\alpha \cap U_\beta \cap U_\gamma$, $\forall\alpha, \beta, \gamma$, we have

$$g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma} \tag{78}$$

This implies that $\omega(\mathscr{C}\ell(\mathcal{M})) = \mathrm{id}_{(2)}$, i.e., M is Cěch trivial and the coherent lifts can be classified by an element of the first Cěch cohomology group $\check{H}^1(M, \mathbb{H}^*)$. Benn and Tucker (1988) proved the following important result:

Theorem. There exists a bundle of irreducible representation modules for $\mathscr{C}\ell(\mathcal{M})$ if and only if the transition maps of $\mathscr{C}\ell(\mathcal{M})$ can be coherently lifted from $\mathrm{Aut}(\mathscr{C}\ell_{1,3})$ to $\mathscr{C}\ell^\star_{1,3}$.

They showed also by defining the concept of equivalence classes of coherent lifts that such classes are in one-to-one correspondence with the equivalence classes of bundles of irreducible representation modules of

[7]Without the $\mathbb{H}$-linear structure there exist more general bundles of irreducible modules for $\mathscr{C}\ell(\mathcal{M})$ (Benn and Tucker, 1988).

$\mathscr{C}\ell(\mathcal{M})$, $I(\mathcal{M})$ and $I'(\mathcal{M})$ being equivalent if there is a bundle isomorphism ρ: $I(\mathcal{M}) \to I'(\mathcal{M})$ such that

$$\rho(a_x q) = a_x \rho(q), \qquad \forall a_x \in \mathscr{C}\ell(T_x^* M), \quad \forall g \in \pi_I^{-1}(x)$$

By defining that a *spin structure* for M is an equivalence class of bundles of irreducible representation modules for $\mathscr{C}\ell(\mathcal{M})$, represented by $I(\mathcal{M})$, Benn and Tucker showed that this agrees with the usual conditions for M to be a spin manifold.

Now, recalling the definition of a vector bundle, we see that the prescription for the construction of $I(\mathcal{M})$ is the following. Let $\{U_\alpha\}$ be an open covering of M with $f_{\alpha\beta}$ being the transition functions for $\mathscr{C}\ell(\mathcal{M})$ and let $\{g_{\alpha\beta}\}$ be a coherent lift, which is then used to quotient the set $\cup_\alpha U_\alpha \times I$, where, e.g., $I = \mathscr{C}\ell_{1,3}\frac{1}{2}(1 + E_0)$ to form the bundle $\cup_\alpha U_\alpha \times I/\mathfrak{R}$, where $\mathfrak{R}$ is the equivalence relation defined as follows. For each $x \in U_\alpha$ we choose a minimal left ideal $I^\alpha_{\Sigma(x)}$ in $\mathscr{C}\ell(T_x^* M)$ by requiring[8]

$$\overset{\Delta}{\varphi}_\alpha(I^\alpha_{\Sigma(x)}) = I \tag{79}$$

As before, we introduce $a_{\alpha\beta} \in \mathscr{C}\ell(T_x^* M)$ such that

$$\overset{\Delta}{\varphi}_\beta(a_{\alpha\beta}) = g_{\alpha\beta}(x) \tag{80}$$

Then for all $X \in \mathscr{C}\ell(T_x^* \mathcal{M})$, $\overset{\Delta}{\varphi}_\alpha(X) = \overset{\Delta}{\varphi}_\beta(a_{\alpha\beta} X a_{\alpha\beta}^{-1})$. So, if $X \in I^\alpha_{\Sigma(x)}$, then $a_{\alpha\beta} X a_{\alpha\beta}^{-1}$ and also $X a_{\alpha\beta}^{-1} \in I^\beta_{\Sigma(x)}$. Putting $Y_\alpha = U_\alpha \times I^\alpha_{\Sigma(x)} Y = \cup_\alpha Y_\alpha$, we define the equivalence relation $\mathfrak{R}$ on Y by $(U_\alpha, x, \psi_\Sigma) \simeq (U_\beta, x, \psi_{\dot\Sigma})$ if and only if

$$\psi_{\dot\Sigma} = \psi_\Sigma a_{\alpha\beta}^{-1} \tag{81}$$

Then, $I(\mathcal{M}) = Y/\mathfrak{R}$ is a bundle which is an irreducible module representation of $\mathscr{C}(\mathcal{M})$. We see that equation (81) captures nicely for $a_{\alpha\beta} \in \mathbf{Spin}_+(1, 3) \subset \mathscr{C}\ell^\star_{1,3}$ our discussion of ADS of Section 2. We then have the following.

Definition. An algebraic Dirac spinor field (ADSF) is a section of $I(\mathcal{M})$ with $a_{\alpha\beta} \in \mathbf{Spin}_+(1, 3) \subset \mathscr{C}\ell^\star_{1,3}$ in equation (81).

From the above results we see that ADSF are equivalence classes of sections of $\mathscr{C}\ell(\mathcal{M})$ and it follows that ADSF can locally be represented by a sum of inhomogeneous differential forms that lie in a minimal left ideal of the Clifford algebra $\mathscr{C}\ell_{1,3}$ at each spacetime point.

In Section 2 we saw that besides the ideal $I = \mathscr{C}\ell_{1,3}\frac{1}{2}(1 + E_0)$, other ideals exist for $\mathscr{C}\ell_{1,3}$ that are only algebraically equivalent to this one. In order to capture all possibilities, we recall that $\mathscr{C}\ell_{1,3}$ can be considered as a

[8]Recall the notation of Section 2, where Σ is an orthonormal frame, etc.

module over itself by left (or right) multiplication by itself. We are thus led to the following.

Definition. The real spin-Clifford bundle of $\mathcal{M}$ is the vector bundle

$$\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) = P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_l \mathcal{C}\ell_{1,3} \tag{82}$$

It is a "principal $\mathcal{C}\ell_{1,3}$ bundle," i.e., it admits a free action of $\mathcal{C}\ell_{1,3}$ on the right (Rodrigues and Oliveira, 1990; Blaine Lawson and Michelson, 1989). There is a natural embedding $P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \subset \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ which comes from the embedding $\mathbf{Spin}_+(1, 3) \subset \mathcal{C}\ell^+_{1,3}$. Hence every real spinor bundle for $\mathcal{M}$ can be captured from $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$. $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ is different from $\mathcal{C}\ell(\mathcal{M})$. Their relation can be discovered by remembering that the representation

$$\text{Ad:} \quad \mathbf{Spin}_+(1, 3) \to \text{Aut}(\mathcal{C}\ell_{1,3}), \qquad \text{Ad}_u X = uXu^{-1}, \qquad u \in \mathbf{Spin}_+(1, 3)$$

is such that Ad_{-1} = identity and so Ad descends to a representation Ad′ of $\text{SO}_+(1, 3)$, which we considered above. It follows that when $P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ exists

$$\mathcal{C}\ell(\mathcal{M}) = P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_{\text{Ad}'} \mathcal{C}\ell_{1,3} \tag{83}$$

From this it is easy to prove that indeed $S(\mathcal{M})$ is a bundle of modules over the bundle of algebras $\mathcal{C}\ell(\mathcal{M})$.

We end this section by defining the local Clifford product of $X \in \sec \mathcal{C}\ell(\mathcal{M})$ by a section of $I(\mathcal{M})$ or $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$. If $\varphi \in I(\mathcal{M})$, we put $X\varphi = \phi \in \sec I(\mathcal{M})$, and the meaning of (83) is that

$$\phi(x) = X(x)\rho(x), \qquad \forall x \in M \tag{84}$$

where $X(x)\varphi(x)$ is the Clifford product of the Clifford numbers $X(x)$, $\varphi(x) \in \mathcal{C}\ell_{1,3}$.

Analogously, if $\psi \in \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$, then

$$X\psi = \xi \in \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \tag{85}$$

and the meaning of equation (84) is the same as in equation (83).

With the above definition we can "identify" from the algebraic point of view sections of $\mathcal{C}\ell(\mathcal{M})$ with sections of $I(\mathcal{M})$ or $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$.

3.3. Dirac–Hestenes Spinor Fields (DHSF)

The main conclusion of Section 3.2 is that a given ADSF which is a section of $I(\mathcal{M})$ can locally be represented by a sum of inhomogeneous differential forms in $\mathcal{C}\ell(\mathcal{M})$ that lies in a minimal left ideal of the Clifford algebra $\mathcal{C}\ell_{1,3}$ at each point $x \in M$. Our objective here is to define a DHSF on $\mathcal{M}$. In order to achieve our goal, we need to find a vector bundle such that a DHSF is an appropriate section.

In Section 2.7 we defined a DHS as an element of the quotient set $\mathcal{C}\ell^+_{1,3}/\mathfrak{R}$, where $\mathfrak{R}$ is the equivalence relation given by equation (50). We immediately realize that if it is possible to define globally on M the equivalence relation $\mathfrak{R}$, then a DHSF can be defined as an even section of the quotient bundle $\mathcal{C}\ell(\mathcal{M})/\mathfrak{R}$.

More precisely, if $\Sigma = \{\gamma^a\}$ $(a = 0, 1, 2, 3)$ and $\dot{\Sigma} = \{\dot{\gamma}^a\}$, $\gamma^a, \dot{\gamma}^a \in \sec \wedge^1(T^*M) \subset \mathcal{C}\ell(M)$ are such that $\dot{\gamma}^a = R\gamma^a R^{-1}$, where $R \in \sec \mathcal{C}\ell^+(\mathcal{M})$ is such that $R(x) \in \mathbf{Spin}_+(1, 3)$ for all $x \in M$, we say that $\dot{\Sigma} \sim \Sigma$. Then a DHSF is an equivalence class of even sections of $\mathcal{C}\ell(\mathcal{M})$ such that its representatives ψ_Σ and $\psi_{\dot{\Sigma}}$ in the basis Σ and $\dot{\Sigma}$ define a set of 1-forms $X^a \in \sec \Lambda^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$ by

$$X^a(x) = \psi_{\dot{\Sigma}}(x)\dot{\gamma}^a(x)\tilde{\psi}_{\dot{\Sigma}}(x) = \psi_\Sigma(x)\gamma^a(x)\tilde{\psi}_\Sigma(x) \tag{86}$$

i.e., ψ_Σ and $\psi_{\dot{\Sigma}}$ are equivalent if and only if

$$\psi_{\dot{\Sigma}} = \psi_\Sigma R^{-1} \tag{87}$$

Observe that for $\dot{\Sigma} \sim \Sigma$ to be globally defined it is necessary that the 1-forms $\{\gamma^a\}$ and $\{\dot{\gamma}^a\}$ are globally defined. It follows that $P_{\mathrm{SO}_+(1,3)}(\mathcal{M})$, the principal bundle of orthonormal frames, must have a global section, i.e., it must be trivial. This conclusion follows directly from our definitions, and it is a necessary condition for the existence of a DHSF. It is obvious that the condition is also sufficient. This suggests the following.

Definition. A spacetime $\mathcal{M}$ admits a spinor structure if and only if it is possible to define a global DHSF on it.

Then, we have the following result.

Theorem. Let $\mathcal{M}$ be a spacetime (dim $M = 4$). Then the necessary and sufficient condition for M to admit a spinor structure is that $P_{\mathrm{SO}_+(1,3)}(\mathcal{M})$ admits a global section.

In Section 3.1 we defined the spinor structure as the principal bundle $P_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ and a theorem with the same statement as the above one is known in the literature as Geroch's (1968) theorem. Geroch's theorem deals with the existence of covariant spinor fields on $\mathcal{M}$, but since we already proved, e.g., that covariant Dirac spinors are equivalent to DHS, our theorem and Geroch's are equivalent. This can be seen more clearly once we verify that

$$\frac{\mathcal{C}\ell(\mathcal{M})}{\mathfrak{R}} \equiv \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \tag{88}$$

where $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) = P_{\mathbf{Spin}_+(1,3)} \times_l \mathcal{C}\ell_{1,3}$ is the spin-Clifford bundle defined in Section 3.1. To see this, recall that a DHSF determines through equation (84) a set of 1-forms $X^a \in \sec \wedge^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$. Under an active transformation,

$$X^a \mapsto \dot{X}^a = RX^aR^{-1}, \qquad R(x) \in \mathbf{Spin}_+(1, 3), \qquad \forall x \in M \tag{89}$$

we obtain the active transformation of a DHSF, which in the Σ-frame is given by[9]

$$\psi_\Sigma \mapsto \psi'_\Sigma = R\psi_\Sigma \tag{90}$$

From equation (87) it follows that the action of $\mathbf{Spin}_+(1, 3)$ on the typical fiber $\mathcal{C}\ell_{1,3}$ of $\mathcal{C}\ell(\mathcal{M})/\mathcal{R}$ must be through left multiplication, i.e., given $u \in \mathbf{Spin}_+(1, 3)$ and $X \in \mathcal{C}\ell_{1,3}$, and taking into account that $\mathcal{C}\ell_{1,3}$ is a module over itself, we can define $l_u \in \text{End}(\mathcal{C}\ell_{1,3})$ by $l_u(X) = ux$, $\forall X \in \mathcal{C}\ell_{1,3}$. In this way we have a representation l: $\mathbf{Spin}_+(1, 3) \to \text{End}(\mathcal{C}\ell_{1,3})$, $u \mapsto l_u$. Then we can write

$$\frac{\mathcal{C}\ell(\mathcal{M})}{\mathcal{R}} = P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_l \mathcal{C}\ell_{1,3}$$

3.4. A Comment on Amorphous Spinor Fields

Crumeyrolle (1991) gives the name of amorphous spinors fields to ideal sections of the Clifford bundle $\mathcal{C}\ell(\mathcal{M})$. Thus an amorphous spinor field ϕ is a section of $\mathcal{C}\ell(\mathcal{M})$ such that $\phi e = \phi$, with e being an idempotent section of $\mathcal{C}\ell(\mathcal{M})$.

It is clear from our discussion of the Fierz identities that are fundamental for the physical interpretation of Dirac theory that these fields cannot be used in a physical theory. The same holds true for the so-called Dirac–Kähler fields (Kähler, 1962; Graf, 1978; Becher, 1981; Hehl and Datta, 1971), which are sections of $\mathcal{C}\ell(\mathcal{M})$. These fields do not have the appropriate transformation law under a Lorentz rotation of the local tetrad field. In particular, the Dirac–Hestenes equation written for amorphous fields is not covariant (see Section 6). We think that with our definitions of algebraic and DH spinor fields physicists can safely use our formalism, which is not only nice, but extremely powerful.

4. THE COVARIANT DERIVATIVE OF CLIFFORD AND DIRAC–HESTENES SPINOR FIELDS

In what follows, as in Section 3, $\mathcal{M} = \langle M, \nabla, g\rangle$ will denote a general Riemann–Cartan spacetime. Since $\mathcal{C}\ell(\mathcal{M}) = \tau M/J(\mathcal{M})$, it is clear that any

[9] Observe also that in the $\dot{\Sigma}$ we have for the representative of the actively transformed DHSF the relation $\psi'_\Sigma = R\psi_\Sigma R^{-1}$.

linear connection defined in τM such that $\nabla g = 0$ passes to the quotient $\tau M/J(\mathcal{M})$ and thus defines an algebra bundle connection (Crumeyrolle, 1991). In this way, the covariant derivative of a Clifford field $A \in \sec \mathcal{C}\ell(\mathcal{M})$ is completely determined.

Although the theory of connections in a principal fiber bundle and on its associate vector bundles is well described in many textbooks, we recall below the main definitions concerning this theory. A full understanding of the various equivalent definitions of a connection is necessary in order to deduce a nice formula that permits us to calculate in a simple way the covariant derivative of Clifford fields and of Dirac–Hestenes spinor fields (Section 4.3). Our simple formula arises due to the fact that the Clifford algebra $\mathcal{C}\ell_{1,3}$, the typical fiber of $\mathcal{C}\ell(\mathcal{M})$, is an associative algebra.

4.1. Parallel Transport and Connections in Principal and Associated Bundles

To define the concept of a connection on a PFB ($\mathbf{P}$, M, π, G) over a four-dimensional manifold M ($\dim G = n$), we first recall that the total space $\mathbf{P}$ of that PFB is itself an $(n + 4)$-dimensional manifold and each one of its fibers $\pi^{-1}(x)$, $x \in M$, is an n-dimensional submanifold of $\mathbf{P}$. The tangent space $T_p\mathbf{P}$, $p \in \pi^{-1}(x)$, is an $(n + 4)$-dimensional linear space and the tangent space $T_p\pi^{-1}(x)$ of the fiber over x, at the same point $p \in \pi^{-1}(x)$, is an n-dimensional linear subspace of $T_p\mathbf{P}$. It is called a *vertical subspace* of $T_p\mathbf{P}$ and is denoted by $V_p\mathbf{P}$.

A connection is a mathematical object that governs the parallel transport of frames along smooth paths in the base manifold M. Such a transport takes place in $\mathbf{P}$, along directions specified by vectors in $T_p\mathbf{P}$, which *does* not lie within the vertical space $V_p\mathbf{P}$. Since the tangent vectors to the paths on the base manifold passing through a given point $x \in M$ span the entire tangent space T_xM, the corresponding vectors $\mathbf{X} \in T_p\mathbf{P}$ (in whose direction parallel transport can generally take place in $\mathbf{P}$) span a four-dimensional linear subspace of $T_p\mathbf{P}$ called a *horizontal space* of $T_p\mathbf{P}$ and denoted by $H_p\mathbf{P}$. The mathematical concept of a connection is given formally by the following.

Definition. A connection on a PFB ($\mathbf{P}$, M, π, G) is a field of vector spaces $H_p\mathbf{P} \subset T_p\mathbf{P}$ such that:

1. π': $H_p\mathbf{P} \to T_xM$, $x = \pi(p)$, is an isomorphism.
2. $H_p\mathbf{P}$ depends differentially on p.
3. $H_{\tilde{R}_g p} = \tilde{R}'_g(H_p)$.

The elements of $H_p\mathbf{P}$ are called *horizontal vectors* and the elements of $T_p\pi^{-1}(x) = V_p\mathbf{P}$ are called *vertical vectors*. In view of the fact that π: $\mathbf{P} \to M$ is a smooth map of the entire manifold $\mathbf{P}$ onto the base manifold M, we have that $\pi' \equiv \pi_*: T\mathbf{P} \to TM$ is a globally defined map from the entire tangent bundle $T\mathbf{P}$ (over the bundle space $\mathbf{P}$) onto the tangent bundle TM.

If $x = \pi(p)$, then due to the fact that $x = \pi(p(t))$ for any curve in $\mathbf{P}$ such that $p(t) \in \pi^{-1}(x)$ and $p(0) = 0$, we conclude that π' maps all vertical vectors into the zero vector in T_xM, that is, $\pi'(V_p\mathbf{P}) = 0$, and we have

$$T_p\mathbf{P} = H_p\mathbf{P} \oplus V_p\mathbf{P}, \qquad p \in \mathbf{P}$$

so that every $\mathbf{X} \in T_p\mathbf{P}$ can be written

$$\mathbf{X} = \mathbf{X}_h + \mathbf{X}_v, \qquad \mathbf{X}_h \in H_p\mathbf{P}, \quad \mathbf{X}_v \in V_p\mathbf{P}$$

Therefore, if $\mathbf{X} \in T_p\mathbf{P}$, we get $\pi'(\mathbf{X}) = \pi'(\mathbf{X}_h) = X \in T_xM$. Then $\mathbf{X}_h$ is called the *horizontal lift* of $X \in T_xM$. An equivalent definition for a connection on $\mathbf{P}$ is given by the following.

Definition. A connection on the principal fiber bundle $(\mathbf{P}, M, \pi, G)$ is a mapping $\Gamma_p: T_xM \to T_pP$, $x = \pi(p)$, such that:

1. Γ_p is linear.
2. $\pi' \circ \Gamma_p = \mathrm{Id}_{T_xM}$, where Id_{T_xM} is the identity mapping in T_xM, and π' is the differential of the canonical projection mapping $\pi: \mathbf{P} \to M$.
3. The mapping $p \mapsto \Gamma_p$ is differentiable.
4. $\Gamma_{R_g p} = R_g' \Gamma_p$, $g \in G$, and R_g being the right translation in $(\mathbf{P}, \pi, M, G)$.

Definition. Let C: $\mathrm{R} \supset I \to M$, $t \mapsto C(t)$, with $x_0 = C(0) \in M$, be a curve in M and let $p_0 \in \mathbf{P}$ be such that $\pi(p_0) = x_0$. The parallel transport of p_0 along C is given by the curve $\mathbf{C}$: $\mathrm{R} \supset I \to \mathbf{P}$, $t \mapsto \mathbf{C}(t)$, defined by

$$\frac{d}{dt}\mathbf{C}(t) = \Gamma_p \frac{d}{dt} C(t)$$

with $\mathbf{C}(0) = p_0$, $\mathbf{C}(t) = p_{\|}$, $\pi(p_{\|}) = x = C(t)$.

We now need to know more about the nature of the vertical space $V_p\mathbf{P}$. For this, let $\hat{X} \in T_eG = \mathfrak{G}$ be an element of the Lie algebra of G and let f: $G \supset U_e \to \mathrm{R}$, where U_e is some neighborhood of the identity element of $\mathfrak{G}$. The vector $\hat{\mathbf{X}}$ can be viewed as the tangent to the curve produced by the exponential map

$$\hat{\mathbf{X}}(f) = \frac{d}{dt} f(\exp(\hat{\mathbf{X}}t))|_{t=0}$$

Then for every $u \in \mathbf{P}$ we can attach to each $\hat{\mathbf{X}} \in T_eG$ a unique element of $V_p\mathbf{P}$ as follows: Let $\mathcal{F}$: $\mathbf{P} \to \mathrm{R}$ be given by

$$\hat{\mathbf{X}}_v(p)(\mathcal{F}) = \frac{d}{dt}\mathcal{F}(p \exp(\hat{\mathbf{X}}t))|_{t=0}$$

By this construction we have attached to each $\hat{\mathbf{X}} \in T_eG$ a unique global section of $T\mathbf{P}$, called the fundamental field corresponding to this element. We then have the canonical isomorphism

$$\hat{\mathbf{X}}_v(p) \leftrightarrow \hat{\mathbf{X}}, \qquad \hat{\mathbf{X}}_v(p) \in V_p\mathbf{P}, \qquad \hat{\mathbf{X}} \in T_eG$$

and we have

$$V_p\mathbf{P} \simeq \mathfrak{G}$$

It follows that another equivalent definition for a connection is as follows.

Definition. A connection on $(\mathbf{P}, M, \pi, G)$ is a 1-form field $\boldsymbol{\omega}$ on $\mathbf{P}$ with values in the Lie algebra $\mathfrak{G}$ such that, for each $p \in \mathbf{P}$, we have:

1. $\boldsymbol{\omega}_p(\mathbf{X}_v) = \hat{\mathbf{X}}$, $\mathbf{X}_v \in V_p\mathbf{P}$, and $\hat{\mathbf{X}} \in \mathfrak{G}$ are related by the canonical isomorphism.
2. $\boldsymbol{\omega}_p$ depends differentially on p.
3. $\boldsymbol{\omega}_{\tilde{R}_g p}(\tilde{R}'_g \mathbf{X}) = (\mathrm{Ad}_{g^{-1}} \omega_p)(\mathbf{X})$.

It follows that if $\{\mathcal{G}_a\}$ is a basis of $\mathfrak{G}$ and $\{\theta^i\}$ is a basis of $T_p^*\mathbf{P}$, we can write $\boldsymbol{\omega}$ as

$$\boldsymbol{\omega}_p = \boldsymbol{\omega}^a \otimes \mathcal{G}_a = \boldsymbol{\omega}_i^a \theta^i \otimes \mathcal{G}_a \tag{91}$$

where $\boldsymbol{\omega}^a$ are 1-forms on $\mathbf{P}$.

The horizontal spaces $H_p\mathbf{P}$ can then be defined by

$$H_p\mathbf{P} = \ker(\boldsymbol{\omega}_p)$$

and we can verify that this is equivalent to the definition of $H_p\mathbf{P}$ given in the first definition of a connection.

Now, for a given connection $\boldsymbol{\omega}$, we can associate with each differentiable local section of $\pi^{-1}(U) \subset \mathbf{P}$, $U \subset M$, a 1-form with values in $\mathfrak{G}$. Indeed, let

$$f\colon \quad M \supset U \to \pi^{-1}(U) \subset \mathbf{P}, \qquad \pi \circ f = \mathrm{Id}_M$$

be a local section of $\mathbf{P}$. We define the 1-form $f^*\boldsymbol{\omega}$ on U with values in $\mathfrak{G}$ by the pullback of $\boldsymbol{\omega}$ by f. If $X \in T_xM$, $x \in U$,

$$(f^*\boldsymbol{\omega})_x(X) = \boldsymbol{\omega}_{f(x)}(f'X)$$

Conversely, we have the following result.

Theorem. Given $\boldsymbol{\omega} \in TM \otimes \mathfrak{G}$ and a differentiable section of $\pi^{-1}(U)$, $U \subset M$, there exists one and only one connection ω on $\pi^{-1}(U)$ such that $f^*\boldsymbol{\omega} = \omega$.

It is important to keep in mind also the following result:

Theorem. On each principal fiber bundle with paracompact base manifold there exist infinitely many connections.

As is well known, each local section f determines a local trivialization

$$\Phi: \quad \pi^{-1}(U) \to U \times G$$

of $\pi: \mathbf{P} \to M$ by setting $\Phi^{-1}(x, g) = f(x)g$. Conversely, Φ determines f, since $f(x) = \Phi^{-1}(x, e)$, where e is the identity of G. We shall also need the following result.

Proposition. Let there be given a local trivialization (U, Φ), $\Phi: \pi^{-1}(U) \to U \times G$, and let $f: M \supset U \to \mathbf{P}$ be the local section associated to it. Then the connection form can be written

$$(\Phi^{-1*}\boldsymbol{\omega})_{x,g} = g^{-1}dg + g^{-1}\omega g \tag{92}$$

where $\boldsymbol{\omega} = f^*\omega \in TU \otimes \mathfrak{G}$. We usually write, by abuse of notation, $\Phi^{-1*}\boldsymbol{\omega} \equiv \boldsymbol{\omega}$. (The proof of this proposition is trivial.)

We can now determine the nature of span$(H_p\mathbf{P})$. Using local coordinates $\langle x^i \rangle$ for $U \subset M$ and g_{ij} for $U_e \in g$,[10] we can write

$$\boldsymbol{\omega} = g_{ij}^{-1}dg_{ij} + g^{-1}\omega g$$

$$\omega = \omega_\mu^A \mathcal{G}_A dx^\mu = \omega^A \otimes \mathcal{G}_A \in T_x U \otimes \mathfrak{G}$$

and

$$[\mathcal{G}_A, \mathcal{G}_B] = f_{ABC}\mathcal{G}_C$$

with f_{ABC} being the structure constants of the Lie algebra $\mathfrak{G}$ of the group G.

Recall now that $\dim H_p\mathbf{P} = 4$. Let its basis be

$$\frac{\partial}{\partial x^\mu} + d_{\mu ij}\frac{\partial}{\partial g_{ij}}$$

$\mu = 0, 1, 2, 3$ and $i, j = 1, \ldots, n = \dim G$. Since $H_p\mathbf{P} = \ker(\boldsymbol{\omega}_p)$, we obtain, by writing

$$\mathbf{X}_h = \beta^\mu\left(\frac{\partial}{\partial x^\mu} + d_{\mu ij}\frac{\partial}{\partial g_{ij}}\right)$$

that

$$d_{\mu ij} = -\omega_\mu^A \mathcal{G}_{Aik} g_{kl}$$

where $\mathcal{G}_{Aik}$ are the matrix elements of $\mathcal{G}_A$.

[10]For simplicity, G is supposed here to be a matrix group. The g_{ij} are then the elements of the matrix representing the element $g \in G$.

Consider now the vector bundle $E = \mathbf{P} \times_{\rho(G)} F$ associated to the PFB $(\mathbf{P}, M, \pi, G)$ through the linear representation ρ of G in the vector space F. Consider the local trivialization $\varphi_\alpha: \pi^{-1}(U_\alpha) \to U_\alpha \times G$ of $(\mathbf{P}, M, \pi, G)$, $\varphi_\alpha(p) = (\pi(p), \overset{\Delta}{\varphi}_\alpha(p))$ with $\overset{\Delta}{\varphi}_{\alpha,x}(p): \pi^{-1}(x) \to G, x \in U_\alpha \in M$. Also, consider the local trivialization $\chi_\alpha: \boldsymbol{\pi}^{-1}(U_\alpha) \to U_\alpha \times F$ of E, where $\overset{\Delta}{\boldsymbol{\pi}}: E \to M$ is the canonical projection. We have $\chi_\alpha(y) = (\boldsymbol{\pi}(Y), \chi_\alpha(y))$ with $\chi_{\alpha,x}(y): \boldsymbol{\pi}^{-1}(x) \to F$. Then, for each $x \in U_{\alpha\beta} = U_\alpha \cap U_\beta$ we must have

$$\overset{\Delta}{\chi}_{\beta,x} \circ \overset{\Delta}{\chi}{}^{-1}_{\beta,x} = \rho(\overset{\Delta}{\varphi}_{\beta,x} \circ \overset{\Delta}{\varphi}{}^{-1}_{\alpha,x})$$

We then have the following.

Definition. The parallel transport of $v_0 \in E$, $\pi(v_0) = x_0$, along the curve $C: \mathbb{R} \supset I \to M$, $x_0 = C(0)$, from x_0 to $x = C(t)$ is the element $v_\| \in E$ such that:

1. $\pi(v_\|) = x$.
2. $\overset{\Delta}{\chi}_{\alpha,x}(v_\|) = \rho(\overset{\Delta}{\varphi}_{\alpha,x}(p_\|) \circ \overset{\Delta}{\varphi}{}^{-1}_{\alpha,x_0}(p_0))\overset{\Delta}{\varphi}_{\beta,x_0}(v_0)$.

Definition. Let X be a vector at $x_0 \in M$ tangent to the curve $C: t \mapsto C(t)$ on M, $x_0 = C(0)$. The covariant derivative of $X \in \sec E$ in the direction of V at x_0 is $(\nabla_V X)_{X_0} \in \sec E$ such that

$$(\nabla_V X)(x_0) \equiv (\nabla_V X)_{x_0} = \lim_{t\to 0} \frac{1}{t} (X^0_{\|,t} - X_0) \tag{93}$$

where $X^0_{\|,t}$ is the "vector" $X_t \equiv X(x(t))$ of a section $X \in \sec E$ parallel transported along C from $x(t)$ to x_0, the unique requirement on C being $(d/dt)C(t)|_{t=0} = V$.

In the local trivialization (U_α, χ_α) of E we have

$$\overset{\Delta}{\chi}_\alpha(X^0_{\|,t}) = \rho(g_0 g_t^{-1})\overset{\Delta}{\chi}_{\alpha,x(t)}(X_t) \tag{94}$$

From this last definition it is trivial to calculate the covariant derivative of $A \in \sec \mathcal{C\ell}(\mathcal{M})$ in the direction of V. Indeed, since a *spin manifold* for M is (Section 3)

$$\mathcal{C\ell}(\mathcal{M}) = P_{SO_+(1,3)} \times_{Ad'} \mathcal{C\ell}_{1,3} = P_{\mathbf{Spin}_+(1,3)} \times_{Ad} \mathcal{C\ell}_{1,3}$$

g_0, $g_t^{-1} \in \mathbf{Spin}_+(1, 3)$, and ρ is the adjoint representation of $\mathbf{Spin}_+(1, 3)$ in $\mathcal{C\ell}_{1,3}$, we can verify (just take into account that our bundle is trivial and put $g_0 = 1$ for simplicity) that we can write

$$A^0_{\|,t} = g_t^{-1} A_t g_t, \qquad g_t = g(x(t)) \in \mathbf{Spin}_+(1, 3) \tag{95}$$

Then,

$$(\nabla_V A)(x_0) = \lim_{t \to 0} \frac{1}{t} (g_t^{-1} A_t g_t - A_0) \tag{96}$$

Now, as we observed in Section 2, each $g \in \mathbf{Spin}_+(1, 3)$ is of the form $\pm e^{F(x)}$, where $F \in \sec \wedge^2(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$, and F can be chosen in such a way as to have a positive sign in this expression, except in the particular case where $F^2 = 0$ and $R = -e^F$. We then write[11]

$$g_t = e^{-1/2\omega t} \tag{97}$$

and

$$\omega = -2g_t' g_t^{-1}|_{t=0} \tag{98}$$

Using equation (98) in equation (97) gives

$$(\nabla_V A)(x_0) = \left\{ \frac{d}{dt} A_t + \frac{1}{2} [\omega, A_t] \right\} \Big|_{t=0} \tag{99}$$

Now let $\langle x^\mu \rangle$ be a coordinate chart for $U \subset M$, $e_a = h_a^\mu \partial_\mu$, $a = 0, 1, 2, 3$, an orthonormal basis for $TU \subset TM$.[12] Let $\gamma^a \in \sec(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$ be the dual basis of $\{e_a\} \equiv \mathcal{B}$. Let $\Sigma = \{\gamma^a\}$ and $\{\gamma_a, a = 0, 1, 2, 3\}$ the reciprocal basis of $\{\gamma^a\}$, i.e., $\gamma^a \cdot \gamma_b = \delta_b^a$, where the dot is the internal product in $\mathcal{C}\ell_{1,3}$. We have $\gamma^a = h_\mu^a dx^\mu$, $\gamma_a = h_a^\mu \eta_{\mu\alpha} dx^\alpha$. We have

$$\nabla_{\partial_\mu} \partial_\nu = \Gamma_{\mu\nu}^\alpha \partial_\alpha, \qquad \nabla_{\partial_\mu}(dx^\alpha) = -\Gamma_{\mu\beta}^\alpha (dx^\beta) \tag{100}$$

$$\nabla_{e_a} e_b = \omega_{ab}^c e_c, \qquad \nabla_{e_a} \gamma^b = -\omega_{ac}^b \gamma^c, \qquad \nabla_{e_a} \gamma_b = \omega_{ab}^c \gamma_c \tag{101}$$

$$\nabla_{e_\mu} e_b = \omega_{\mu b}^c e_c, \qquad \nabla_\mu \gamma^b = -\omega_{\mu c}^b \gamma^c, \qquad \nabla_\mu \gamma_b = \omega_{\mu b}^c \gamma_c \tag{102}$$

From equation (100) we easily obtain ($\nabla_{\partial_\mu} \equiv \nabla_\mu$)

$$(\nabla_\mu A) = \partial_\mu A + \tfrac{1}{2}[\omega_\mu, A] \tag{103}$$

with

$$\omega_\mu = -2(\partial_\mu g) g^{-1} \in \sec \overset{2}{\wedge}(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M}) \tag{104}$$

where $g \in \sec \mathcal{C}\ell^+(\mathcal{M})$ is such that $g|_{c(t)} \equiv g_t \in \mathbf{Spin}_+(1, 3)$.

We observe that formulas (100) and (101) for the covariant derivative of a homogeneous Clifford field preserves (as it must) its graduation, i.e., if

[11] The negative sign in the definition of ω is only for convenience, in order to obtain formulas in agreement with known results.

[12] Since M is a spin manifold, $P_{SO_+(1,3)}(\mathcal{M})$ is trivial and $\{e_a\}$, $a = 0, 1, 2, 3$, can be taken as a global tetrad field for the tangent bundle.

$A_p \in \sec \wedge^p(T^*M) \subset \sec \mathscr{C}\ell(\mathcal{M})$, $p = 0, 1, 2, 3, 4$, then $[\omega_\mu, A_p] \in \sec \wedge^p(T^*M) \subset \sec \mathscr{C}\ell(\mathcal{M})$, as can be easily verified.

Since

$$\tfrac{1}{2}[\omega_\mu, \gamma^\alpha] = \omega_\mu \cdot \gamma^\alpha = -\gamma^\alpha \cdot \omega_\mu \tag{105}$$

we have

$$\omega_\mu = \tfrac{1}{2}\omega_\mu^{ab}(\gamma_a \wedge \gamma_b) \tag{106}$$

and we observe that

$$\omega_\mu^{ab} = -\omega_\mu^{ba} \tag{107}$$

For $A = A_a\gamma^a$ we immediately obtain

$$\nabla_{e_a} A_b = e_a(A_b) - \omega_{ab}^c A_c \tag{108}$$

which agrees with the well-known formula for the derivative of a covariant vector field.

Also, we have

$$\nabla_\mu A_a = \partial_\mu(A_a) - \omega_{\mu a}^b A_b$$
$$\nabla_\mu A_\alpha = \partial_\mu(A_\alpha) - \Gamma_{\mu\alpha}^\beta A_\beta \tag{109}$$

From the general formula (99) the next result follows immediately:

Proposition. The covariant derivative ∇_X on $\mathscr{C}\ell(\mathcal{M})$ acts as a derivation on the algebra of sections, i.e., for $A, B \in \sec \mathscr{C}\ell(\mathcal{M})$ we have

$$\nabla_X(AB) = (\nabla_X A)B + A(\nabla_X B) \tag{110}$$

The proof is trivial.

4.2. The Lie Derivative of Clifford Fields

Let $V \in \sec TM$ be a vector field on M which induces a local one-parameter transformation group $t \mapsto \varphi_t$. If φ_{*t} stands as usual for the natural extension of the tangent map $d\varphi_t$ to tensor fields, the Lie derivative $\pounds_V$ of a given tensor field $X \in \sec TM$ is defined by

$$(\pounds_V X)(x) = \lim_{t\to 0} \frac{1}{t}(X_x - (\varphi_{*t}(x))_x) \tag{111}$$

$\pounds_V$ is a derivation in the tensor algebra $\tau\mathcal{M}$. Then we have, for $a, b \in \sec \wedge^1(T^*\mathcal{M}) \subset \mathscr{C}\ell(\mathcal{M})$,

$$\pounds_V(a \otimes b + b \otimes a - 2g^{-1}(a, b))$$
$$= (\pounds_V a) \otimes b + b \otimes (\pounds_V a) - 2\pounds_V(g^{-1}(a, b)) \tag{112}$$

Since $a \otimes b + b \otimes a - 2g^{-1}(a, b)$ belongs to $J(\mathcal{M})$, the bilateral ideal generating the Clifford bundle $\mathcal{C}\ell(\mathcal{M})$, we see from (111) that $\pounds_V$ preservers $J(\mathcal{M})$ if and only if $\pounds_V g = 0$, i.e., V induces a local isometry group, and then V is a Killing vector (Choquet-Bruhat *et al.*, 1982).

4.3. The Covariant Derivative of Algebraic Dirac Spinor Fields

As discussed in Section 3, ADSF are sections of the real spinor bundle $I(\mathcal{M}) = P_{\mathrm{Spin}_+(1,3)}(M) \times_l I$, where $I = \mathcal{C}\ell_{1,3}\frac{1}{2}(1 + E_0)$. Here $I(\mathcal{M})$ is a subbundle of the spin-Clifford bundle $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$. Since both $I(\mathcal{M})$ and $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$ are vector bundles, the covariant derivatives of ADSF or DHSF can be immediately calculated using the general method discussed in Section 4.1.

Before we calculate the covariant spinor derivative ∇_V^s of a section of $I(\mathcal{M})$ [or $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$], where $V \in \sec TM$ is a vector field, we must recall that ∇_V^s is a module derivation (Blaine Lawson and Michelson, 1989), i.e., if $X \in \sec \mathcal{C}\ell(\mathcal{M})$ and $\varphi \in \sec I(\mathcal{M})$ [or $\sec \mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$], then the following holds:

Proposition. Let ∇ be the connection in $\mathcal{C}\ell(\mathcal{M})$ to which ∇^s is related. Then,

$$\nabla_V^s(X\varphi) = (\nabla_V X)\varphi + X(\nabla_V^s\varphi) \tag{113}$$

The proof of this proposition is trivial once we derive an explicit formula to compute $\nabla_V^s(\varphi)$, $\varphi \in \sec I(\mathcal{M}) \subset \sec \mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$.

Let us now calculate the covariant derivative ∇_v^s in the direction of v, a vector at $x_0 \in M$ of $\phi \in \sec I(\mathcal{M}) \subset \sec \mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$.

Putting $g_0 = 1 \in \mathbf{Spin}_+(1, 3)$, we have, using the general procedure,

$$\phi_{\|,t}^0 = g_t^{-1}\phi_t \tag{114}$$

where $\phi_{\|,t}^0$ is the "vector" $\phi_t = \phi(x(t))$ of a section $\phi \in \sec I(\mathcal{M}) \subset \sec \mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$ parallel transported along C: $\mathrm{R} \supset I \rightarrow M$, $t \mapsto C(t)$, from $x(t) \equiv C(t)$ to $x_0 = C(0)$, $(d/dt)C(t)|_{t=0} = v$.

Putting as in equation (98) $g_t = e^{-1/2\omega t}$, we get, by using equation (94),

$$(\nabla_v^s\phi)(x_0) = \left(\frac{d}{dt}\phi_t + \frac{1}{2}\omega\phi_t\right)\Bigg|_{t=0} \tag{115}$$

If $\{\gamma^a\}$ is an orthogonal field of 1-forms, $\gamma^a \in \sec \wedge^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$ dual to the orthogonal frame field $\{e_a\}$, $e_a \in \sec TM$, $g(e_a, e_b) = \eta_{ab}$, and if $\{\gamma_a\}$ is the reciprocal frame of $\{\gamma^a\}$, i.e., $\gamma^a \cdot \gamma_b = \delta_b^a$ $(a, b = 0, 1, 2, 3)$, then for equation (115) we get

$$\nabla_{e_a}^s\phi = e_a(\phi) + \tfrac{1}{2}\omega_a\phi \tag{116}$$

with

$$\omega_a = \tfrac{1}{2}\omega_a^{bc}\gamma_b \wedge \gamma_c \tag{117}$$

and we recognize the 1-forms ω_a as being $\omega_a = \omega(e_a)$, where $\omega = f^*\boldsymbol{\omega}$, f: $M \to U \times G$, is the global section used to write (114). The Lie algebra of $\mathbf{Spin}_+(1, 3)$ is, of course, generated by the "vectors" $\{\gamma_a \wedge \gamma_b\}$. We have

$$\nabla_{e_a}\gamma^b = -\omega_a^{bc}\gamma_c \tag{118}$$

If $\langle x^\mu \rangle$ is a coordinate chart for $U \subset M$ and $\gamma^a = h_\mu^a dx^\mu$, a, $\mu = 0, 1, 2, 3$, we also obtain

$$\nabla_\mu^s\phi = \partial_\mu(\phi) + \tfrac{1}{2}\omega_\mu\phi, \qquad \omega_\mu = \tfrac{1}{2}\omega_\mu^{bc}\gamma_b \wedge \gamma_c \tag{119}$$

Now, since $\phi \in \sec I(\mathcal{M}) \subset \sec \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ is such that $\phi e_\Sigma = \phi$ with $e_\Sigma = \frac{1}{2}(1 + \gamma^0)$, it follows from $\nabla_{e_a}^s\phi = \nabla_{e_a}^s(\phi e_\Sigma)$ that

$$e_\Sigma \nabla_{e_a}^s e_\Sigma = 0 \tag{120}$$

Now, recalling equation (30), we have a spinorial basis for $I(\mathcal{M})$ given by $\beta^s = \{s^A\}$, $A = 1, 2, 3, 4$, $s^A \in \sec I(\mathcal{M})$, with

$$s^1 = e_\Sigma = \tfrac{1}{2}(1 + \gamma^0), \qquad s^2 = -\gamma^1\gamma^3 e_\Sigma, \qquad s^3 = \gamma^3\gamma^0 e_\Sigma, \qquad s^4 = \gamma^1\gamma^0 e_\Sigma \tag{121}$$

Then, as we learned in Section 2, $\phi = \phi_A s^A$, where ϕ_A are formally complex numbers. Then

$$\begin{aligned} \nabla_{e_a}^s\phi &= e_a(\phi) + \tfrac{1}{2}\omega_a\phi \\ &= [e_a(\phi_A) + \tfrac{1}{2}\omega_a\phi_A]s^A \\ &= (e_a(\phi_A) + \tfrac{1}{2}[\omega_a]_A^B\phi_B)s^A \end{aligned} \tag{122}$$

with

$$\omega_a s^A = [\omega_a]_B^A s^B \tag{123}$$

$$\begin{aligned} \nabla_{e_a}^s\phi &= \nabla_{e_a}^s(\phi_A s^A) \\ &= e_a(\phi_A)s^A + \phi_A\nabla_{e_a}^s s^A \end{aligned} \tag{124}$$

From equations (122) and (124) it follows that

$$\nabla_{e_a}^s s^A = \tfrac{1}{2}[\omega_a]_B^A s^B \tag{125}$$

We introduce the dual space $I^*(\mathcal{M})$ of $I(M)$, where $I^*(M) = P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_r I$, where here the action of $\mathbf{Spin}_+(1, 3)$ on the typical fiber is on the right. A basis for $I^*(\mathcal{M})$ is then $\rho_s = \{s_A\}$, $A = 1, 2, 3, 4$, $s_A \in \sec I^*(\mathcal{M})$, such that

$$s_A(s^B) = \delta_A^B \tag{126}$$

A simple calculation shows that

$$\nabla^s_{e_a} s_A = -\tfrac{1}{2}[\omega_a]^B_A s_B \tag{127}$$

Since $\mathscr{C}\ell(\mathcal{M}) = I^*(\mathcal{M}) \otimes I(\mathcal{M})$ (the "tensor-spinor space") is spanned by the basis $\{s^A \otimes s_B\}$, we can write

$$\gamma_a s_A = [\gamma_a]^B_A s_B \tag{128}$$

with

$$[\gamma_a]^B_A = \gamma^B_{aA} \equiv \gamma_a(s^B, s_A) \tag{129}$$

being the matricial representation of γ_a. It follows that

$$\nabla^s_{e_b}\gamma_a(s^B, s_A) = e_b([\gamma_a]^B_A) - \omega^c_{ba}\gamma^B_{CA} + \tfrac{1}{2}\omega^B_{bC}\gamma^C_{bA} - \tfrac{1}{2}\omega^C_{bA}\gamma^B_{aC} \tag{130}$$

Now,

$$(\tfrac{1}{2}\omega^B_{bc}\gamma^C_{aA} - \tfrac{1}{2}\omega^C_{bA}\gamma^B_{aC})s^A = (\gamma_a \cdot \omega_b)s^B \tag{131}$$

and from $\omega_b = \frac{1}{2}\omega_b^{cd}\gamma_c \wedge \gamma_d$, we get

$$(\gamma_a \cdot \omega_b)s^B = (-\omega^c_{ba}\gamma^B_{cA})s^A \tag{132}$$

From equations (131) and (132) we obtain

$$\tfrac{1}{2}\omega^B_{bC}\gamma^C_{aA} - \tfrac{1}{2}\omega^C_{bA}\gamma^B_{aC} = -\omega^d_{ba}\gamma^B_{dA} \tag{133}$$

and then

$$\nabla^s_{e_b}[\gamma_a]^B_A = e_b([\gamma_a]^B_A) = 0 \tag{134}$$

since, according to a result obtained in Section 2.6, $[\gamma_a]^B_C$ are constant matrices. Equation (133) agrees with the result presented, e.g., in Choquet-Bruhat *et al.* (1982). Also, from $\omega_a = \frac{1}{2}\omega_a^{bc}\gamma_b \wedge \gamma_c$ it follows that

$$\omega^A_{aB} = \tfrac{1}{2}\omega_a^{bc}[\gamma_b, \gamma_c]^A_B \tag{135}$$

We can also easily obtain the following results: Writing

$$\nabla^s_{e_a}\phi \equiv (\nabla^s_{e_a}\phi_A)s^A \tag{136}$$

it follows that

$$\nabla^s_{e_a}\phi_A = e_a(\phi_A) + \tfrac{1}{8}\omega^b_{ac}[\gamma_b, \gamma^c]^B_A\phi_B \tag{137}$$

and

$$\nabla^s_{e_a}\phi^A = e_a(\phi^A) - \tfrac{1}{8}\omega^b_{ac}[\gamma_b, \gamma^c]^A_B\phi^B \tag{138}$$

Equation (138) agrees exactly with the result presented, e.g., by Choquet-Bruhat *et al.* (1982) for the components of the covariant derivative of a CDSF $\psi \in \sec P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_\rho \mathbb{C}^4$. It is important to emphasize here that the condition given by (134), namely $\nabla^s_{e_b}[\gamma_a]^B_A = 0$ holds true, but this does not imply that $\nabla_{e_b}\gamma^a = 0$, i.e., ∇ need not be the so-called connection of parallelization of the $\mathcal{M} = \langle M, g, \nabla\rangle$, which, as is well known, has zero curvature but nonzero torsion (Bishop and Goldberg, 1980).

The main difference between ∇^s acting on sections of $I(\mathcal{M})$ or of $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ and ∇ acting on sections of $\mathcal{C}\ell(\mathcal{M})$ is that, for $\phi \in \sec I(\mathcal{M})$ or $\sec \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ and $A \in \sec \mathcal{C}\ell(\mathcal{M})$, we must have

$$\nabla^s_{e_a}(A\phi) = (\nabla_{e_a}A)\phi + A(\nabla^s_{e_a}\phi) \tag{139}$$

and of course ∇ cannot be applied to sections of $I(\mathcal{M})$ or of $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$.

4.4. The Representative of the Covariant Derivative of a Dirac–Hestenes Spinor Field in $\mathcal{C}\ell(\mathcal{M})$

In Section 3.2 we defined a DHSF ψ as an even section of $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$. Then, by the same procedure used in Section 4.3, we get[13]

$$\nabla^s_{e_a}\psi = e_a(\psi) + \tfrac{1}{2}\omega_a\psi, \qquad \nabla^s_{e_a}\tilde{\psi} = e_a(\tilde{\psi}) - \tfrac{1}{2}\tilde{\psi}\omega_a \tag{140}$$

and as before

$$\omega_a = \tfrac{1}{2}\omega_a^{bc}\gamma_b \wedge \gamma_c \in \sec \mathcal{C}\ell(\mathcal{M}) \tag{141}$$

Now, let $\gamma^a \in \sec \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ such that $\gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}$ $(a, b = 0, 1, 2, 3)$, and let us calculate $\nabla^s_{e_a}(\psi\gamma^b)$. Using equation (116), we have

$$\nabla^s_{e_a}(\psi\gamma^b) = e_a(\psi\gamma^b) + \tfrac{1}{2}\omega_a\psi\gamma^b = (\nabla^s_{e_a}\psi)\gamma^b \tag{142}$$

On the other hand,

$$\nabla^s_{e_a}(\psi\gamma^b) = (\nabla^s_{e_a}\psi)\gamma^b + \psi(\nabla^s_{e_a}\gamma^b) \tag{143}$$

Comparison of equations (142) and (143) implies that

$$\nabla^s_{e_a}\gamma^b = 0 \tag{144}$$

The matrix version of equation (144) is equation (134).

We know that if $\psi, \tilde{\psi} \in \sec \mathcal{C}\ell^+_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$, then $\psi\gamma^a\tilde{\psi} = \mathbf{X}^a$ is such that $\mathbf{X}^a(x) \in \mathbb{R}^{1,3}$, $\forall x \in M$. Then,

$$\nabla^s_{e_a}(\psi\gamma^b\tilde{\psi}) = (\nabla^s_{e_a}\psi)\gamma^b\tilde{\psi} + \psi\gamma^b(\nabla^s_{e_a}\tilde{\psi}) \tag{145}$$

and $\nabla^s_{e_a}(\psi\gamma^b\tilde{\psi})(x) \in \mathbb{R}^{1,3}$, $\forall x \in M$.

[13] The meaning of e_a, γ^b, etc., is as before.

We are now prepared to find the representative of the covariant derivative of a DHSF in $\mathcal{C}\ell(\mathcal{M})$. We recall that ψ is an equivalence class of even sections of $\mathcal{C}\ell(\mathcal{M})$ such that in the basis $\Sigma = \{\gamma^a\}$, $\gamma^a \in \sec \wedge^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$, the representative of ψ is $\psi_\Sigma \in \mathcal{C}\ell^+(\mathcal{M})$ and the representative of $\mathbf{X}^a$ is $X^a \in \sec \wedge^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$ such that

$$X^a = \psi_\Sigma \gamma^a \tilde{\psi}_\Sigma \tag{146}$$

Let ∇ be the connection acting on sections of $\mathcal{C}\ell(\mathcal{M})$. Then,

$$\begin{aligned}\nabla_{e_a}(\psi_\Sigma \gamma^b \tilde{\psi}_\Sigma) &= \{e_a(\psi_\Sigma) + \tfrac{1}{2}[\omega_a, \psi_\Sigma]\}\gamma^b\tilde{\psi}_\Sigma \\ &\quad + \psi_\Sigma(\nabla_{e_a}\gamma^b)\tilde{\psi}_\Sigma + \psi_\Sigma\gamma^b\{e_a(\psi_\Sigma) + \tfrac{1}{2}[\omega_a, \tilde{\psi}_\Sigma]\} \\ &= [e_a(\psi_\Sigma) + \tfrac{1}{2}\omega_a\psi_\Sigma]\gamma^b\psi_\Sigma + \psi_\Sigma\gamma^b[e_a(\psi_\Sigma) - \tfrac{1}{2}\tilde{\psi}_\Sigma\omega_a]\end{aligned} \tag{147}$$

Comparing equations (145) and (147), we see that the following definition suggests itself.

Definition:

$$\begin{aligned}(\nabla^s_{e_a}\psi)_\Sigma &\equiv \nabla^s_{e_a}\psi_\Sigma = e_a(\psi_\Sigma) + \tfrac{1}{2}\omega_a\psi_\Sigma \\ (\nabla^s_{e_a}\tilde{\psi})_\Sigma &\equiv \nabla^s_{e_a}\tilde{\psi}_\Sigma = e_a(\tilde{\psi}_\Sigma) - \tfrac{1}{2}\tilde{\psi}_\Sigma\omega_a \\ (\nabla^s_{e_a}\gamma^b)_\Sigma &\equiv \nabla^s_{e_a}\gamma^b = 0\end{aligned} \tag{148}$$

where $(\nabla^s_{e_a}\psi)_\Sigma$, $(\nabla^s_{e_a}\tilde{\psi})_\Sigma$, $(\nabla^s_{e_a}\gamma^b)_\Sigma \in \sec \mathcal{C}\ell(\mathcal{M})$ are representatives of $\nabla^s_{e_a}\psi$ (etc.) in the basis Σ in $\mathcal{C}\ell(\mathcal{M})$.

Observe that the result $\nabla^s_{e_a}\gamma^b = 0$ is compatible with the result $\nabla^s_{e_a}[\gamma_a]^B_A = 0$ obtained in equation (133) and is an important result in order to write the Dirac–Hestenes equation (Section 6).

5. THE FORM DERIVATIVE OF THE MANIFOLD AND THE DIRAC AND SPIN-DIRAC OPERATORS

Let $\mathcal{M} = \langle M, g, \nabla\rangle$ be a Riemann–Cartan manifold (Section 4), and let $\mathcal{C}\ell(\mathcal{M})$, $I(\mathcal{M})$, and $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$ be respectively the Clifford, real spinor, and spin-Clifford bundles. Let ∇^s be the spinorial connection acting on sections of $I(\mathcal{M})$ or $\mathcal{C}\ell_{\mathrm{Spin}_+(1,3)}(\mathcal{M})$. Let also $\{e_a\}$, $\{\gamma^a\}$ have the same meaning as before and for convenience when useful we shall denote the Pfaff derivative by $\partial_a \equiv e_a$.

Definition. Let Γ be a section of $\mathcal{C}\ell(\mathcal{M})$, $I(\mathcal{M})$, or $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$. The form derivative of the manifold is a canonical first-order differential operator $\partial: \Gamma \mapsto \Gamma$ such that

$$\begin{aligned} \partial\Gamma &= (\gamma^a \partial_a)\Gamma \\ &= \gamma^a \cdot (\partial_a(\Gamma)) + \gamma^a \wedge (\partial_a(\Gamma)) \end{aligned} \tag{149}$$

for $\gamma^a \in \sec \mathcal{C}\ell(\mathcal{M})$.

Definition. The Dirac operator acting on sections of $\mathcal{C}\ell(\mathcal{M})$ is a canonical first-order differential operator $\boldsymbol{\partial}$: $A \mapsto \boldsymbol{\partial} A$, $A \in \sec \mathcal{C}\ell(\mathcal{M})$, such that

$$\boldsymbol{\partial} A = (\gamma^a \nabla_{e_a})A = \gamma^a \cdot (\nabla_{e_a} A) + \gamma^a \wedge (\nabla_{e_a} A) \tag{150}$$

Definition. The spin-Dirac operator[14] acting on sections of $I(\mathcal{M})$ of $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$ is a canonical first-order differential operator $\mathbf{D}$: $\Gamma \to \mathbf{D}\Gamma$ [$\Gamma \in \sec I(\mathcal{M})$] [or $\Gamma \in \sec \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$] such that

$$\begin{aligned} \mathbf{D}\Gamma &= (\gamma^a \nabla^s_{e_a})\Gamma \\ &= \gamma^a \cdot (\nabla^s_{e_a}\Gamma) + \gamma^a \wedge (\nabla^s_{e_a}\Gamma) \end{aligned} \tag{151}$$

The operator $\boldsymbol{\partial}$ is sometimes called the Dirac–Kahler operator when $\mathcal{M}$ is a Lorentzian manifold (Graf, 1978), i.e., $\mathbf{T}(\nabla) = 0$, $\mathbf{R}(\nabla) = 0$, where $\mathbf{T}$ and $\mathbf{R}$ are respectively the torsion and Riemann tensors. In this case we can show that

$$\boldsymbol{\partial} = d - \delta \tag{152}$$

where d is the differential operator and δ the Hodge codifferential operator. In the spirit of Section 4, we use the convention that the representative of $\mathbf{D}$ [acting on sections of $\mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$] in $\mathcal{C}\ell(\mathcal{M})$ also will be denoted by

$$\mathbf{D} = \gamma^a \nabla^s_{e_a} \tag{153}$$

6. THE DIRAC–HESTENES EQUATION IN MINKOWSKI SPACETIME

Let $\mathcal{M} = \langle M, g, \nabla \rangle$ be the Minkowski spacetime, $\mathcal{C}\ell(\mathcal{M})$ be the Clifford bundle of $\mathcal{M}$ with typical fiber $\mathcal{C}\ell_{1,3}$, and let $\Psi \in \sec P_{\mathbf{Spin}_+(1,3)}(\mathcal{M}) \times_\rho \mathbb{C}^4$ [with ρ the $D^{(1/2,0)} \oplus D^{(0,1/2)}$ representation of $SL(2, \mathbb{C}) \simeq \mathbf{Spin}_+(1, 3)$]. Then,

[14] In Blaine Lawson and Michelson (1989) this operator [acting on sections of $I(\mathcal{M})$] is called simply the Dirac operator, being the generalization of the operator originally introduced by Dirac. See also Benn and Tucker (1987) for comments on the use of this terminology.

the Dirac equation for the charged fermion field Ψ in interaction with the electromagnetic field A is (Bjorken and Drell, 1964) ($\hbar = c = 1$)

$$\underline{\gamma}^\mu(i\partial_\mu - eA_\mu)\Psi = m\Psi \qquad \text{or} \qquad i\mathbf{D}\psi - \underline{\gamma}^\mu A_\mu \Psi = m\Psi \tag{154}$$

where $\underline{\gamma}^\mu\underline{\gamma}^\nu + \underline{\gamma}^\nu\underline{\gamma}^\mu = 2\eta^{\mu\nu}$, $\underline{\gamma}^\mu$ being the Dirac matrices given by (31), and $A = A_\mu dx^\mu \in \sec \wedge^1(T^*M)$.

As shown, e.g., in Rodrigues and Oliveira (1990), this equation is equivalent to the following equation satisfied by $\phi \in \sec I(\mathcal{M})$ [$\phi e_\Sigma = \phi$, $e_\Sigma = \frac{1}{2}(1 + \gamma^0)$, $\boldsymbol{\gamma}^\mu\boldsymbol{\gamma}^\nu + \boldsymbol{\gamma}^\nu\boldsymbol{\gamma}^\mu = 2\eta^{\mu\nu}$, $\boldsymbol{\gamma}^\mu \in \sec \mathcal{C}\ell_{\mathbf{Spin}_+(1,3)}(\mathcal{M})$]:

$$\mathbf{D}\phi\gamma^2\gamma^1 - eA\phi = m\phi \tag{155}$$

where $\mathbf{D}$ is the Dirac operator on $I(\mathcal{M})$ and $A \in \sec \wedge^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$.

Since, as discussed in Section 3, each ϕ is an equivalence class of sections of $\mathcal{C}\ell(\mathcal{M})$, we can also write an equation equivalent to (155) for $\phi_\Sigma = \phi_\Sigma e_\Sigma$, ϕ_Σ, $e_\Sigma \in \sec \mathcal{C}\ell(\mathcal{M})$, $e_\Sigma = \frac{1}{2}(1 + \gamma^0)$, $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$, $\gamma^\mu \in \sec \mathcal{C}\ell(\mathcal{M})$, and $\gamma^\mu = dx^\mu$ for the global coordinate functions $\langle x^\mu \rangle$. In this case the Dirac operator $\boldsymbol{\partial} = \gamma^\mu\nabla_\mu$ is equal to the form derivative $\partial = \gamma^\mu\partial_\mu$ and we have

$$\partial\phi_\Sigma\gamma^2\gamma^1 - eA\phi_\Sigma = m\phi_\Sigma\gamma^0 \tag{156}$$

Since each ϕ_Σ can be written $\phi_\Sigma = \psi_\Sigma e_\Sigma$ [$\psi_\Sigma \in \sec \mathcal{C}\ell^+(\mathcal{M})$ being the representative of a DHSF] and $\gamma^0 e_\Sigma = e_\Sigma$, we can write the following equation for ψ_Σ, which is equivalent to the Dirac equation (Rodrigues and Oliveira, 1990; Lounesto, 1993, 1994)

$$\partial\psi_\Sigma\gamma^2\gamma^1 - eA\psi_\Sigma = m\psi_\Sigma\gamma^0 \tag{157}$$

which is the so-called Dirac–Hestenes equation (Hestenes, 1967, 1976).

Equation (157) is covariant under passive (and active) Lorentz transformations, in the following sense: consider the change from the Lorentz frame $\Sigma = \{\gamma^\mu = dx^\mu\}$ to the frame $\dot\Sigma = \{\dot\gamma^\mu = d\dot x^\mu\}$ with $\dot\gamma^\mu = R^{-1}\gamma^\mu R$ and $R \in \mathbf{Spin}_+(1, 3)$ being constant. Then the representative of the Dirac–Hestenes spinor changes, as discussed in Section 3, from ψ_Σ to $\psi_{\dot\Sigma} = \psi_\Sigma R^{-1}$. Then we have $\partial = \gamma^\mu\partial_\mu = \dot\gamma^\mu\partial/\partial\dot x^\mu$, where $\langle x^\mu \rangle$ and $\langle \dot x^\mu \rangle$ are related by a Lorentz transformation and

$$\partial\psi_\Sigma R^{-1}R\gamma^2R^{-1}R\gamma^1R^{-1} - eA\psi_\Sigma R^{-1} = m\psi_\Sigma R^{-1}R\gamma^0R^{-1} \tag{158}$$

i.e.,

$$\partial\psi_{\dot\Sigma}\dot\gamma^2\dot\gamma^1 - eA\psi_{\dot\Sigma} = m\psi_{\dot\Sigma}\dot\gamma^0 \tag{159}$$

Thus our definition of the Dirac–Hestenes spinor fields as an equivalence class of even sections of $\mathcal{C}\ell(\mathcal{M})$ solves directly the question raised by Parra (1992) concerning the covariance of the Dirac–Hestenes equation.

Observe that if ∇^s is the spinor covariant derivative acting on ψ_Σ (defined in Section 4.4), we can write equation (157) in intrinsic form, i.e., without the need of introducing a chart for $\mathcal{M}$, as follows:

$$\gamma^a \nabla^s_a \psi_\Sigma \gamma^2 \gamma^1 - eA\psi_\Sigma = m\psi_\Sigma \gamma^0 \tag{160}$$

where γ^a is now an orthogonal basis of T^*M, and it is not necessarily that $\gamma^a = dx^a$ for some coordinate functions x^a.

It is well known that equation (154) can be derived from the principle of stationary action through variation of the action

$$S(\Psi) = \int d^4x\, \mathcal{L} \tag{161}$$

$$\mathcal{L} = -\frac{i}{2}(\gamma^\mu \partial_\mu \Psi^+)\Psi + \frac{i}{2}\Psi^+(\underline{\gamma}^\mu \partial_\mu \overline{\Psi}) - m\Psi^+\overline{\Psi} - eA_\mu \Psi^+ \underline{\gamma}_\mu \overline{\Psi} \tag{162}$$

with $\Psi^+ = \Psi^* \gamma^0$.

In the next section we present the rudiments of the multiform derivative approach to Lagrangian field theory (MDALFT) developed in Choquet-Bruhat *et al.* (1982); see also Lasenby *et al.*, 1993) and apply this formalism to obtain the Dirac–Hestenes equation on a Riemann–Cartan spacetime.

7. LAGRANGIAN FORMALISM FOR THE DIRAC–HESTENES SPINOR FIELD ON A RIEMANN–CARTAN SPACETIME

In this section we apply the concept of multiform (or multivector) derivatives first introduced by Hestenes and Sobczyk (1984) (HS) to present a Lagrangian formalism for the Dirac–Hestenes spinor field DHSF on a Riemann–Cartan spacetime. In Section 7.1 we briefly present our version of the multiform derivative approach to Lagrangian field theory for a Clifford field $\phi \in \sec \mathcal{C}\ell(\mathcal{M})$, where $\mathcal{M}$ is Minkowski spacetime. In Section 7.2 we present the theory for the DHSF on Riemann–Cartan spacetime.

7.1. Multiform Derivative Approach to Lagrangian Field Theory

We define a Lagrangian density for $\phi \in \sec \mathcal{C}\ell(\mathcal{M})$ as a mapping

$$\mathcal{L}\colon\ (x, \phi(x), \boldsymbol{\partial} \wedge \phi(x), \boldsymbol{\partial} \cdot \phi(x)) \mapsto \mathcal{L}(x, \phi(x), \boldsymbol{\partial} \wedge \phi(x), \boldsymbol{\partial} \cdot \phi(x)) \in \overset{4}{\wedge}(T^*M) \subset \mathcal{C}\ell(\mathcal{M}) \tag{163}$$

where $\boldsymbol{\partial}$ is the Dirac operator acting on sections of[15] $\mathscr{C}\ell(\mathcal{M})$, and by the above notation we mean an arbitrary multiform function of ϕ, $\boldsymbol{\partial} \wedge \phi$, and $\boldsymbol{\partial} \cdot \phi$.

In this section we perform our calculations using an orthonormal and coordinate basis for the tangent (and cotangent) bundle. If $\langle x^\mu \rangle$ is a global Lorentz chart, then $\gamma^\mu = dx^\mu$ and $\boldsymbol{\partial} = \gamma^\mu \nabla_\mu = \gamma^\mu \partial_\mu = \partial$, so that the Dirac operator ($\boldsymbol{\partial}$) coincides with the form derivative (∂) of the manifold.

We introduce also for ϕ a Lagrangian

$$L(x, \phi(x), \partial \wedge \phi(x), \partial \cdot \phi(x)) \in \wedge^0(T^*M) \subset \mathscr{C}\ell(\mathcal{M})$$

by

$$\mathscr{L}(x, \phi(x), \partial \wedge \phi(x), \partial \cdot \phi(x)) = L(x, \phi(x), \partial \wedge \phi(x), \partial \cdot \phi(x))\tau_g \quad (164)$$

where $\tau_g \subset \sec \wedge^4(T^*M)$ is the volume form, $\tau_g = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$ for $\langle x^\mu \rangle$ a global Lorentz chart.

In what follows we suppose that $\mathscr{L}[L]$ does not depend explicitly on x and we write $L(\phi, \partial \wedge \phi, \partial \cdot \phi)$ for the Lagrangian. Observe that

$$L(\phi, \partial \wedge \phi, \partial \cdot \phi) = \langle L(\phi, \partial \wedge \phi, \partial \cdot \phi)\rangle_0 \quad (165)$$

As usual, we define the action for ϕ as

$$S(\phi) = \int_U L(\phi, \partial \wedge \phi, \partial \cdot \phi)\tau_g, \qquad U \subseteq M \quad (166)$$

The field equation for ϕ is obtained from the principle of stationary action for $S(\phi)$. Let $\eta \in \sec \mathscr{C}\ell(\mathcal{M})$ containing the same grades as $\phi \in \sec \mathscr{C}\ell(\mathcal{M})$. We say that ϕ is stationary with respect to L if

$$\frac{d}{dt} S(\phi + t\eta)|_{t=0} = 0 \quad (167)$$

But, recalling Hestenes and Sobczyk (1984), we see that equation (167) is just the definition of the multiform derivative of $S(\phi)$ in the direction of η, i.e., we have, using the notation of HS,

$$\eta * \partial_\phi S(\phi) = \frac{d}{dt} S(\phi + t\eta)|_{t=0} \quad (168)$$

[15] An example of a Lagrangian of the form given by equation (163) appears, e.g., in the theory of the gravitational field in Minkowski spacetime (Rodrigues and de Souza, 1993). In de Souza and Rodrigues (1994) we present further mathematical results derived in the Clifford bundle formalism. Those results are important for the gravitational theory and other field theories.

Then

$$\frac{d}{dt} S(\phi + t\eta)|_{t=0} = \int \tau_g \frac{d}{dt} \{L[(\phi + t\eta), \partial \wedge (\phi + t\eta), \partial \cdot (\phi + t\eta)]\}|_{t=0} \tag{169}$$

Now,

$$\frac{d}{dt} \{[L(\phi + t\eta), \partial \wedge (\phi + t\eta), \partial \cdot (\phi + t\eta)]\}_{t=0}$$

$$= \eta * \partial_{\phi} L + (\partial \wedge \eta) * \partial_{\partial\wedge\phi} L + (\partial \cdot \eta) * \partial_{\partial\cdot\phi} L \tag{170}$$

Before we calculate (170) for a general $\phi \in \sec \mathcal{C}\ell(\mathcal{M})$, let us suppose that $\phi = \langle\phi\rangle_r$, i.e., it is homogeneous. Using the properties of the multiform derivative (Hestenes and Sobcyzk, 1984), we obtain after some algebra the following fundamental formulas ($\eta = \langle\eta\rangle_r$):

$$\eta * \partial_{\phi_r} L = \eta \cdot \partial_{\phi_r} L \tag{171}$$

$$(\partial \wedge \eta) * \partial_{\partial\wedge\phi_r} L = \partial \cdot [\eta \cdot (\partial_{\partial\wedge\phi_r} L)] - (-1)^r \eta \cdot [\partial \cdot (\partial_{\partial\wedge\phi_r} L)] \tag{172}$$

$$(\partial \cdot \eta) * \partial_{\partial\cdot\phi_r} L = \partial \cdot [\eta \cdot (\partial_{\partial\cdot\phi_r} L)] + (-1)^r \eta \cdot [\partial \wedge (\partial_{\partial\cdot\phi_r} L)] \tag{173}$$

Inserting equation (7.9) into (170) and then in equation (169), we obtain, imposing $(d/dt)S(\phi_r + t\eta) = 0$,

$$\int_U \{\eta \cdot [\partial_{\phi_r} L - (-1)^r \partial \cdot (\partial_{\partial\wedge\phi_r} L) + (-1)^r \partial \wedge (\partial_{\partial\cdot\phi_r} L)]\}\tau_g + \int_U \partial \cdot [\eta \cdot (\partial_{\partial\wedge\phi_r} L + \partial_{\partial\cdot\phi_r} L)]\tau_g = 0 \tag{174}$$

The last integral in (174) is null by Stokes' theorem if we suppose as usual that η vanishes on the boundary of U.

Then equation (174) reduces to

$$\int_U \{\eta \cdot [\partial_{\phi_r} L - (-1)^r \partial \cdot (\partial_{\partial\wedge\phi_r} L) + (-1)^r \partial \wedge (\partial_{\partial\cdot\phi_r} L)]\}\tau_g = 0 \tag{175}$$

Now since $\eta = \langle\eta\rangle_r$ is arbitrary and $\partial_{\phi_r} L$, $\partial \cdot (\partial_{\partial\wedge\phi_r} L)$, and $\partial \wedge (\partial_{\partial\cdot\phi_r} L)$ are of grade r, we get

$$\langle \partial_{\phi_r} L - (-1)^r \partial \cdot (\partial_{\partial\wedge\phi_r} L) + (-1)^r (\partial_{\partial\cdot\phi_r} L)\rangle_r = 0 \tag{176}$$

But since $\partial_{\phi_r}\langle L\rangle_0 = \langle\partial_{\phi_r} L\rangle_r = \partial_{\phi_r} L$, $\partial_{\partial\wedge\phi_r} L = \langle\partial_{\partial\wedge\phi_r} L\rangle_{r+1}$, etc., equation (176) reduces to

$$\partial_{\phi_r} L - (-1)^r \partial \cdot (\partial_{\partial\wedge\phi_r} L) + (-1)^r \partial \wedge (\partial_{\partial\cdot\phi_r} L) = 0 \tag{177}$$

Equation (177) is a *multiform Euler–Lagrange equation*. Observe that as $L = \langle L\rangle_0$ the equation has the graduation of $\phi_r \in \sec \wedge^r(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$.

Now, let $X \in \sec \mathcal{C}\ell(\mathcal{M})$ be such that $X = \Sigma_{s=0}^{4}\langle X\rangle_r$ and $F(x) = \langle F(x)\rangle_0$. From the properties of the multivectorial derivative we can easily obtain

$$\begin{aligned}\partial_X F(x) &= \partial_X\langle F(x)\rangle_0 \\ &= \sum_{s=0}^{4} \partial_{\langle X\rangle_s}\langle F(x)\rangle_0 = \sum_{s=0}^{4} \langle \partial_{\langle X\rangle_s} F(X)\rangle_0 \end{aligned} \tag{178}$$

In view of this result, if $\phi = \Sigma_{r=0}^{4}\langle\phi\rangle_r \in \sec \mathcal{C}\ell(\mathcal{M})$, we get as *Euler–Lagrange equation* for ϕ the following equation:

$$\sum_r [\partial_{\langle\phi\rangle_r} L - (-1)^r \partial\cdot(\partial_{\partial\wedge\langle\phi\rangle_r} L) + (-1)^r \partial \wedge (\partial_{\partial\cdot\langle\phi\rangle_r} L)] = 0 \tag{179}$$

We can write equations (177) and (179) in a more convenient form if we take into account that $A_r \cdot B_s = (-1)^{r(s-1)} B_s \cdot A_r$ $(r \leq s)$ and $A_r \wedge B_s = (-1)^{rs} B_s \wedge A_r$. Indeed, we now have for ϕ_r that

$$\partial\cdot(\partial_{\partial\wedge\phi_r} L) \equiv \partial\cdot(\partial_{\partial\wedge\phi_r} L)_{r+1} = (-1)^r (\partial_{\partial\wedge\phi_r} L)_{r+1} \cdot \overleftarrow{\partial} \tag{180}$$

$$\partial \wedge (\partial_{\partial\cdot\phi_r} L) \equiv \partial \wedge (\partial_{\partial\cdot\phi_r} L)_{r-1} = (-1)^r (\partial_{\partial\cdot\phi_r} L)_{r+1} \wedge \overleftarrow{\partial} \tag{181}$$

where $\overleftarrow{\partial}$ means that the internal and exterior products are to be done on the right. Then, equation (179) can be written as

$$\partial_\phi L - (\partial_{\partial\wedge\phi} L) \cdot \overleftarrow{\partial} - (\partial_{\partial\cdot\phi} L) \wedge \overleftarrow{\partial} = 0 \tag{182}$$

We now analyze the particular and important case where

$$L(\phi, \partial\wedge\phi, \partial\cdot\phi) = L(\phi, \partial\wedge\phi + \partial\cdot\phi) = L(\phi, \partial\phi) \tag{183}$$

We can easily verify that

$$\partial_{\partial\cdot\phi} L(\partial\phi) = \langle \partial_{\partial\phi} L(\partial\phi)\rangle_{r-1} \tag{184}$$

$$\partial_{\partial\wedge\phi} L(\partial\phi) = \langle \partial_{\partial\phi} L(\partial\phi)\rangle_{r+1} \tag{185}$$

Then, equation (182) can be written

$$\begin{aligned} &\partial_\phi L - \langle \partial_{\partial\phi} L\rangle_{r+1} \cdot \overleftarrow{\partial} - \langle \partial_{\partial\phi} L\rangle_{r-1} \rfloor\, \overleftarrow{\partial} \\ &= \partial_\phi L - \langle (\partial_{\partial_\phi} L)\cdot \overleftarrow{\partial}\rangle_r - \langle (\partial_{\partial\phi} L) \wedge \overleftarrow{\partial}\rangle_r \\ &= \langle \partial_\phi L - (\partial_\phi L)\cdot \overleftarrow{\partial} - (\partial_{\partial\phi} L) \wedge \overleftarrow{\partial}\rangle_r = 0 \\ &= \langle \partial_\phi L - (\partial_\phi L)\, \overleftarrow{\partial}\rangle_r = 0 \end{aligned} \tag{186}$$

whence follows the very elegant equation

$$\partial_\phi L - (\partial_{\partial\phi} L) \overleftarrow{\partial} = 0 \tag{187}$$

also obtained in Lasenby *et al.* (1993).

As an example of the use of equation (187) we write the Lagrangian in Minkowski space for a Dirac–Hestenes spinor field represented in the frame $\Sigma = \{\gamma^\mu\}$ $[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}, \gamma^\mu \in \sec \wedge^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})]$ by $\psi \in \sec \mathcal{C}\ell(\mathcal{M})^+$ in interaction with the electromagnetic field $A \in \sec \wedge^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M})$. We have[16]

$$L = L_{\rm DH} = \langle(\partial\psi\gamma^2\gamma^1 - m\psi\gamma^0)\gamma^0\tilde{\psi} - eA\psi\gamma^0\tilde{\psi}\rangle_0 \tag{188}$$

Then

$$\partial_{\tilde{\psi}} L = (\partial\psi\gamma^2\gamma^1 - m\psi\gamma^0)\gamma_0 - eA\psi\gamma^0 \qquad \text{and} \qquad \partial_{\partial\tilde{\psi}} L = 0 \tag{189}$$

and we get the Dirac–Hestenes equation

$$\partial\psi\gamma^2\gamma^1 - eA\psi = m\psi\gamma^0 \tag{190}$$

Also, since $\langle A\psi\gamma^0\tilde{\psi}\rangle_0 = \langle\psi\gamma^0\tilde{\psi}A\rangle_0$, we have

$$\partial_\psi L = -m\tilde{\psi} - e\gamma^0\tilde{\psi}A \tag{191}$$

$$\partial_{\partial\psi} L = \gamma^{210}\tilde{\psi} \qquad (\gamma^{210} = \gamma^2\gamma^1\gamma^0) \tag{192}$$

Now,

$$(\partial_{\partial\psi} L) \overleftarrow{\partial} = (\gamma^{210}\tilde{\psi}) \overleftarrow{\partial}$$

and from the above equations we get

$$-m\tilde{\psi} - e\gamma^0\tilde{\psi}A - (\gamma^{210}\tilde{\psi}) \overleftarrow{\partial} = 0$$

and this gives again

$$\partial\psi\gamma^2\gamma^1 - eA\psi = m\psi\gamma^0$$

Another Lagrangian that also gives the DH equation is, as can be easily verified,

$$L'_{\rm DH} = \langle\tfrac{1}{2}\partial\psi\gamma^{210}\tilde{\psi} - \tfrac{1}{2}\psi\gamma^{210}\tilde{\psi} \overleftarrow{\partial} - m\psi\tilde{\psi} - eA\psi\gamma^0\psi\rangle_0 \tag{193}$$

7.2. The Dirac–Hestenes Equation on a Riemann–Cartan Spacetime

Let $\mathcal{M} = \langle M, g, \nabla\rangle$ be a Riemann–Cartan spacetime (RCST), i.e., $\nabla g = 0$, $\mathbf{T}(\nabla) \neq 0$, $\mathbf{R}(\nabla) \neq 0$. Let $\mathcal{C}\ell(\mathcal{M})$ be the Clifford bundle of spacetime

[16]Note that we are omitting, for the sake of simplicity, the reference to the basis Σ in the notation for ψ.

with typical fiber $\mathcal{C}\ell_{1,3}$ and let $\psi \in \sec \mathcal{C}\ell^+(\mathcal{M})$ be the representative of a Dirac–Hestenes spinor field in the basis $\Sigma = \{\gamma^a\}$ $[\gamma^a \in \sec \wedge^1(T^*M) \subset \sec \mathcal{C}\ell(\mathcal{M}), \gamma^a\gamma^b + \gamma^b\gamma^a = 2\eta^{ab}]$ dual to the basis $\mathcal{B} = \{e_a\}$, $e_a \in \sec TM$, $a, b = 0, 1, 2, 3$.

To describe the "interaction" of the DHSF ψ with the Riemann–Cartan spacetime we invoke the principle of minimal coupling. This consists in changing $\partial = \gamma^a\partial_a$ in the Lagrangian given by equation (193) by

$$\gamma^a\partial_a\psi \mapsto \gamma^a\nabla^s_{e_a}\psi \tag{194}$$

where $\nabla^s_{e_a}$ is the spinor covariant derivative of the DHSF introduced in Section 4.4, i.e.,

$$\nabla^s_{e_a}\psi = e_a(\psi) + \tfrac{1}{2}\omega_a\psi \tag{195}$$

Let $\langle x^\mu \rangle$ be a chart for $U \subset M$ and let $\partial_a \equiv e_a = h^\mu_a\partial_\mu$ and $\gamma^a = h^a_\mu dx^\mu$, with $h^a_\mu h^\nu_a = \delta^\mu_\nu$, $h^a_\mu h^\mu_b = \delta^a_b$.

We take as the action for the DHSF ψ on a RCST

$$S(\psi) = \int_U \left\langle \frac{1}{2}\,\mathbf{D}\psi\gamma^{210}\tilde{\psi} - \frac{1}{2}\,\psi\gamma^{210}\tilde{\psi}\,\overleftarrow{\mathbf{D}} - m\psi\tilde{\psi} \right\rangle_0 h^{-1}dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{196}$$

where $\mathbf{D} = \gamma^a\nabla^s_{e_a}$ is the Dirac operator made with the spinor connection acting on sections of $\mathcal{C}\ell(\mathcal{M})$ and $h^{-1} = [\det(h^\mu_a)]^{-1}$. The Lagrangian $L = \langle L \rangle_0$ is then

$$\begin{aligned} L &= h^{-1}\left\langle \frac{1}{2}\,\mathbf{D}\psi\gamma^{210}\tilde{\psi} - \frac{1}{2}\,\psi\gamma^{210}\tilde{\psi}\,\overleftarrow{\mathbf{D}} - m\psi\tilde{\psi} \right\rangle_0 \\ &= h^{-1}\left\langle \frac{1}{2}\left[\gamma^a\left(\partial_a + \frac{1}{2}\,\omega_a\psi\right)\gamma^{210}\tilde{\psi} - \psi\gamma^{210}\left(\partial_a\tilde{\psi} - \frac{1}{2}\,\tilde{\psi}\omega_a\right)\gamma^a\right] - m\psi\tilde{\psi} \right\rangle_0 \end{aligned} \tag{197}$$

As in Section 7.2, the principle of stationary action gives

$$\begin{aligned} \partial_{\tilde{\psi}}L - (\partial_{\partial\tilde{\psi}}L)\,\overleftarrow{\partial} &= 0 \\ \partial_\psi L - (\partial_{\partial\psi}L)\,\overleftarrow{\partial} &= 0 \end{aligned} \tag{198}$$

To obtain the equations of motion we must recall that

$$(\partial_{\partial\psi}L)\,\overleftarrow{\partial} = \partial_\mu(\partial_{\partial_\mu\psi}L) \tag{199}$$

and

$$\partial_{\partial_\mu\psi} L = h_a^\mu \partial_{\partial_a\psi} L \tag{200}$$

Then (198) become

$$\begin{aligned} &\partial_\psi L - \partial_\mu(h_a^\mu)\partial_{\partial_a\psi} L - \partial_a(\partial_{\partial_a\psi} L) = 0 \\ &\partial_{\bar\psi} L - \partial_\mu(h_a^\mu)\partial_{\partial_a\bar\psi} L - \partial_a(\partial_{\partial_a\bar\psi} L) = 0 \end{aligned} \tag{201}$$

Now, taking into account that $[e_a, e_b] = c_{ab}^d e_d$ and that $\partial_a h/h = h_\mu^a \partial_a h_a^\mu$, we get

$$\partial_\mu h_a^\mu = -c_{ab}^b + \partial_a \ln h \tag{202}$$

and (201) become

$$\begin{aligned} &\partial_\psi L - [\partial_a + \partial_a \ln h - c_{ab}^b]\partial_{\partial_a\psi} L = 0 \\ &\partial_{\bar\psi} L - [\partial_a + \partial_a \ln h - c_{ab}^b]\partial_{\partial_a\bar\psi} L = 0 \end{aligned} \tag{203}$$

Let us calculate explicitly the second of equations (201). We have

$$\partial_{\bar\psi} = h^{-1}\left[\frac{1}{2}\,\gamma^a(\nabla_{e_a}\psi)\gamma^{210} + \frac{1}{4}\,\omega_a\gamma^a\psi\gamma^{210} - m\psi\right] \tag{204}$$

$$\partial_{\partial_a\bar\psi} L = h^{-1}\left(-\frac{1}{2}\,\gamma^a\psi\gamma^{210}\right) \tag{205}$$

Then,

$$\begin{aligned} \partial_a(\partial_{\partial_a\bar\psi} L) &= (\partial_a \ln h^{-1})h^{-1}\left(-\frac{1}{2}\,\gamma^a\psi\gamma^{210}\right) - h^{-1}\,\frac{1}{2}\,\gamma^a\partial_a\psi\gamma^{210} \\ &= -(\partial_a \ln h)\partial_{\partial_a\bar\psi} L - h^{-1}\,\frac{1}{2}\,\gamma^a\partial_a\psi\gamma^{210} \end{aligned} \tag{206}$$

Using (202) and (204) in the second of equations (201), we obtain

$$\tfrac{1}{2}(\mathbf{D}\psi)\gamma^{210} + \tfrac{1}{4}\omega_a\gamma^a\psi\gamma^{210} - m\psi + \tfrac{1}{2}\gamma^a\partial_a\gamma^{210} - \tfrac{1}{2}c_{ab}^b\gamma^a\psi\gamma^{210} = 0$$

or

$$\mathbf{D}\psi\gamma^{210} - \tfrac{1}{4}(\gamma^a\omega_a - \omega_a\gamma^a)\psi\gamma^{210} - m\psi - \tfrac{1}{2}c_{ab}^b\gamma^a\psi\gamma^{210} = 0$$

Then

$$\mathbf{D}\psi\gamma^{210} - \tfrac{1}{2}(\gamma^a\cdot\omega_a)\psi\gamma^{210} - \tfrac{1}{2}c_{ab}^b\gamma^a\psi\gamma^{210} - m\psi = 0 \tag{207}$$

But

$$\gamma^a \cdot \omega_a = \omega^b_{ba}\gamma^a \tag{208}$$

and since $\omega^b_{ab} = 0$ because $\omega^{bc}_a = -\omega^{cb}_a$, we have

$$\gamma^a \cdot \omega_a = (\omega^b_{ba} - \omega^b_{ab})\gamma^a \tag{209}$$

Using equation (209) in equation (207), we obtain

$$\mathbf{D}\psi\gamma^{210} - \tfrac{1}{2}[\omega^b_{ba} - \omega^b_{ab} + c^b_{ab}]\gamma^a\psi\gamma^{210} - m\psi = 0$$

Recalling the definition of the torsion tensor, $T^c_{ab} = \omega^c_{ba} - \omega^c_{ba} + c^c_{ab}$, we get

$$(\mathbf{D} + \tfrac{1}{2}T)\psi\gamma^1\gamma^2 + m\psi\gamma^0 = 0 \tag{210}$$

where $T = T^b_{ab}\gamma^a$.

Equation (210) is the Dirac–Hestenes equation on Riemann–Cartan spacetime. Observe that if $\mathcal{M}$ is a Lorentzian spacetime [$\nabla g = 0$, $\mathbf{T}(\nabla) = 0$, $\mathbf{R}(\nabla) \neq 0$], then equation (210) reduces to

$$\gamma^a(\partial_a + \tfrac{1}{2}\omega_a)\psi\gamma^1\gamma^2 + m\psi\gamma^0 = 0 \tag{211}$$

which is exactly the equation proposed by Hestenes (1985) as the equation for a spinor field in a gravitational field modeled as a Lorentzian spacetime $\mathcal{M}$. Also, equation (210) is the representation in $\mathcal{C}\ell(\mathcal{M})$ of the spinor equation proposed by Hehl and Datta (1971) for a covariant Dirac spinor field $\Psi \in P_{\text{Spin}_+(1,3)} \times_\rho \mathbb{C}^4$ on a Riemann–Cartan spacetime. The proof of this last statement is trivial. Indeed, first we multiply ψ in (210) by the idempotent field $\frac{1}{2}(1 + \gamma^0)$, thereby obtaining an equation for the representative of the Dirac algebraic spinor field in $\mathcal{C}\ell(\mathcal{M})$. Then we translate the equation in $I(\mathcal{M}) = P_{\text{Spin}_+(1,3)} \times_l I$, whence, taking a matrix representation with the techniques discussed in Section 2, we obtain as equation for $\Psi \in P_{\text{Spin}_+(1,3)} \times_\rho \mathbb{C}^4$,

$$i(\underline{\gamma}_a\nabla^s_a\Psi - \tfrac{1}{2}T\Psi) - m\Psi = 0, \qquad i = \sqrt{-1} \tag{212}$$

with $T = T^b_{ab}\underline{\gamma}^a$, $\underline{\gamma}^a$ being the Dirac matrices [equation (31)].

We comment here that equation (210) looks like, but it is indeed very different from an equation proposed by Ivanenko and Obukhov (1985) as a generalization of the so-called Dirac–Kähler (–Ivanenko) equation for a Riemann–Cartan spacetime. The main differences between the equation given in Ivanenko and Obukhov (1985) and our equation (210) is that in Ivanenko and Obukhov (1985) $\Psi \in \sec \mathcal{C}\ell(\mathcal{M})$, whereas in our approach $\psi_a \in \mathcal{C}\ell^+(\mathcal{M})$ is only the representative of the Dirac–Hestenes spinor field in the basis $\Sigma = \{\gamma^a\}$, and also Ivanenko and Obukhov (1985) use ∇_{e_a} instead of $\nabla^s_{e_a}$.

Finally we comment that equation (210) has played an important role in our recent approach to a geometrical equivalence of the Dirac and Maxwell equations (Vaz and Rodrigues, 1993a) and also in the double solution interpretation of quantum mechanics (Vaz and Rodrigues, 1993a,b; Rodrigues *et al.*, 1993b).

8. CONCLUSIONS

We have presented a rigorous study of the Dirac–Hestenes spinor fields (DHSF), their covariant derivatives, and the Dirac–Hestenes equations on a Riemann–Cartan manifold $\mathcal{M}$.

Our study shows in a definitive way that covariant spinor fields (CDSF) can be represented by DHSF that are equivalence classes of even sections of the Clifford bundle $\mathcal{C}\ell(\mathcal{M})$, i.e., spinors are equivalence classes of a sum of even differential forms. We clarified many misconceptions and misunderstanding in the earlier literature concerned with the representation of spinor fields by differential forms. In particular, we proved that the so-called Dirac–Kähler spinor fields that are sections of $\mathcal{C}\ell(\mathcal{M})$ and are examples of amorphous spinor fields (Section 4.3) cannot be used for the representation of the field of fermionic matter. With amorphous spinor fields the Dirac–Hestenes equation is not covariant.

We have also presented an elegant and concise formulation of Lagrangian theory in the Clifford bundle and used this powerful method to derive the Dirac–Hestenes equation on a Riemann–Cartan spacetime.

ACKNOWLEDGMENTS

The authors are grateful to CNPq and CAPES for financial support.

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